Midterm Examination
February 26, 2015

1. Dynamic Behavior and Stability Analysis of the Moon

Consider a simple, two dimensional model of the moon's orbit, which focuses exclusively on the effect of the Earth on the moon and neglects the effect of the moon on the Earth. In this model, the Earth is stationary at a position of $(x_E, y_E) = (0,0)$. This motion is described by two second-order non-linear differential equations,

$$a_x = \frac{d^2 x_M}{dt^2} = \frac{F_x}{m_M} = -g\frac{m_M m_E}{m_M}\frac{1}{r^3}x_M = -\frac{gm_E}{r^3}x_M$$

$$a_y = \frac{d^2 y_M}{dt^2} = \frac{F_y}{m_M} = -g\frac{m_M m_E}{m_M}\frac{1}{r^3}y_M = -\frac{gm_E}{r^3}y_M$$

where m_M and m_E are the mass of the moon and the earth respectively, x_M and y_M are the coordinates of the position of the moon with respect to Earth, g is the gravitational constant, and r is the radial separation between the earth and moon, $r = \sqrt{x_M^2 + y_M^2}$.

We shall work this problem in reduced units, where time is measure in years, length is measured in units of the distance between the moon and Earth, e.g. (r *should* be 1), $m_M = 1$, $m_E = 81.28$ moon units and g = 85.8596.

For parts (a) through (f) of the problem, consider the following initial conditions, at time t = 0,

$x_M = 1$, $y_M = 0$, $\frac{dx_M}{dt} = 0$ and $\frac{dy_M}{dt} = 83.926$ earth-moon distances/year.

(a) Is this system of ODEs linear or nonlinear?
(b) Convert the system of second order ODES to a system of first order ODEs.
(c) Numerically solve for the dynamic behavior of the moon for 1 year.
(d) According to your model, about how many times does the moon orbit the Earth each year?
(e) Construct a phase plot of x_M and y_M and visually estimate the coordinates of the critical point.
(f) Based on your observed trajectories, describe the type and stability of this critical point. (You do not need to calculate eigenvalues during the exam.)
(g) Holding everything constant except the initial velocity of the moon in the y-direction, solve the problem for a year again for a slower moon with a velocity $\frac{dy_M}{dt} = 52.8$.
(h) What happened in part (g)?
(i) Holding everything constant except the initial velocity of the moon in the y-direction, solve the problem for a year again for a faster moon with a velocity $\frac{dy_M}{dt} = 100.0$.
(j) What happened in part (i)?