Midterm Examination
Administered: Friday, October 18, 2002

Problem (1)
Consider the first order linear ordinary differential equation.

$$\frac{dy}{dt} = f(y,t) = \sin(t)y + 2t \tag{1}$$

with the initial condition

$$y(t_o = 0) = 1 \tag{2}$$

The second order numerical method to solve this problem is given by

$$y_i = y_{i-1} + (t_i - t_{i-1})\frac{1}{2}\left[f(y_{i-1}, t_{i-1}) + f(y_i, t_i)\right] \tag{3}$$

(a) Use Heun's method to approximate y at $t = 0.1$. (Use one interval of size Δt=0.1)
(b) Solve part (a) again but take advantage of the linearity of the ODE to avoid the approximation inherent in Heun's method.
(c) Explain why the answers in (a) and (b) are different? Which answer is more accurate?

Solution

(a) Use Heun's method to approximate y at $t = 0.1$. (Use one interval of size Δt=0.1)

$$\frac{dy}{dt} = f(y, t = 0) = \sin(0)(1) + 2(0) = 0$$

Use Euler method to approximate y at new t:

$$y(t = 0.1) \approx y(t = 0) + \Delta t f(y, t = 0) = 1 + 0.1 \cdot 0 = 1$$

$$\frac{dy}{dt} = f(y = 1, t = 0.1) \approx \sin(0.1)(1) + 2(0.1) = 0.29983341664683$$

$$y(t = 0.1) \approx y(t = 0) + \Delta t \frac{1}{2}\left[f(y = 1, t = 0.0) + f(y = 1, t = 0.1)\right]$$

$$y(t = 0.1) \approx 1 + (0.1)\frac{1}{2}\left[0 + 0.2998\right] = 1.014992$$

(b) Solve part (a) again but take advantage of the linearity of the ODE to avoid the approximation inherent in Heun's method.

$$y_i = y_{i-1} + (t_i - t_{i-1})\frac{1}{2}\left[\sin(t_{i-1})y_{i-1} + 2t_{i-1} + \sin(t_i)y_i + 2t_i\right]$$

$$y_i\left[1-(t_i-t_{i-1})\frac{1}{2}\sin(t_i)\right]=y_{i-1}+(t_i-t_{i-1})\frac{1}{2}\left[\sin(t_{i-1})y_{i-1}+2t_{i-1}+2t_i\right]$$

$$y_i=\frac{y_{i-1}+(t_i-t_{i-1})\frac{1}{2}\left[\sin(t_{i-1})y_{i-1}+2t_{i-1}+2t_i\right]}{1-(t_i-t_{i-1})\frac{1}{2}\sin(t_i)}$$

$$y_i=\frac{1+(0.1)\frac{1}{2}\left[\sin(0)(1)+2(0)+2(0.1)\right]}{1-(0.1)\frac{1}{2}\sin(0.1)}=\frac{1.01}{0.99500832916766}=1.015067$$

(c) Explain why the answers in (a) and (b) are different? Which answer is more accurate?

Part (a) uses the Euler method to approximate the new value of y. Part (b) avoids this approximation because the equation is linear. Part (b) is therefore for accurate.

Problem (2)

Consider the system of linear algebraic equations:

$$\begin{aligned} x_1+x_2 &= 2 \\ 5x_1-6x_2 &= -1 \end{aligned}$$

(a) Demonstrate that the multivariate Newton-Raphson will exactly solve a system of linear algebraic equations in one iteration. Use the initial guess of your choice.

Solution:

The multivariate Newton-Raphson will exactly solve a system of linear algebraic equations in one iteration regardless of the initial guess. Let's use a guess of (0,0).

(a) If we are going to solve this system using multivariate Newton-Raphson, we need the Jacobian and the residual. Determine them.

$$\underline{\underline{J}}=\begin{bmatrix}1 & 1\\ 5 & -6\end{bmatrix} \qquad \underline{R}=\begin{bmatrix}x_1+x_2-2\\ 5x_1-6x_2+1\end{bmatrix}=\begin{bmatrix}-2\\ 1\end{bmatrix}$$

$$\det(\underline{\underline{J}})=(1)(-6)-(5)(1)=-11$$

$$\underline{\underline{J}}^{-1}=\frac{1}{\det(\underline{\underline{J}})}\begin{bmatrix}-6 & -1\\ -5 & 1\end{bmatrix}=\begin{bmatrix}\frac{6}{11} & \frac{1}{11}\\ \frac{5}{11} & -\frac{1}{11}\end{bmatrix}$$

$$\underline{\delta x} = -\underline{\underline{J}}^{-1}\underline{R} = -\begin{bmatrix} \frac{6}{11} & \frac{1}{11} \\ \frac{5}{11} & -\frac{1}{11} \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\underline{x}^{j+1} = \underline{x}^{j} + \underline{\delta x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

We can verify that this is the correct solution, by substituting into the original set of algebraic equations.

Problem (3)

Consider the system of two linear ODES.

$$\frac{dx_1}{dt} = x_1 + x_2$$
$$\frac{dx_2}{dt} = 5x_1 - 6x_2$$

Determine the type of critical point and the stability.

Solution:

In order to determine the type of critical point and the stability we need the eigenvalues.

$$\underline{\underline{A}} - \lambda\underline{\underline{I}} = \begin{bmatrix} 1-\lambda & 1 \\ 5 & -6-\lambda \end{bmatrix}$$

$$(1-\lambda)(-6-\lambda) - 5 = 0$$

$$\lambda^2 + 5\lambda - 11 = 0$$

$$\lambda = \frac{-5 \pm \sqrt{5^2 - 4\cdot 1\cdot(-11)}}{2\cdot 1} = \frac{-5 \pm \sqrt{69}}{2}$$

From here we can see that both eigenvalues are real. One eigenvalue is positive and the other is negative. Therefore, our critical point will be a saddle point. All saddle points are unstable.