Homework Assignment Number Three Solutions

Problem 1.

A chemical plant produces A thousands of liters of "A-plus Liquid Fungicide" and B thousand of liters of "B-Gone Liquid Insecticide" per month. The two processes share some raw materials and facilities so that the amount of A and B produced are not independent of each other. In fact the amount of B produced is related to the amount of A produced by

$$B = RM - \frac{A}{2} + 40$$

where RM is the amount of raw materials received at the plant in a given month (also in liters). The total amount of product in thousands of liters is given as

$$T(A,B) = B + A$$

The monthly production schedule for 2012 is as followed

Month	A (thousand of liters)	RM (thousands of liters)
Jan	50	120
Feb	50	120
Mar	60	110
Apr	70	120
May	80	130
Jun	90	130
Jul	100	130
Aug	100	130
Sep	90	130
Oct	80	120
Nov	70	110
Dec	60	100

In all problems: PUT UNITS WITH ANSWERS OR YOU WILL NOT RECEIVE FULL CREDIT.
In all relevant problems: WRITE DOWN THE FORMULA YOU USE, BEFORE YOU USE IT.

(a) Is this problem continuous or discrete?

The problem is discrete. There are 12 elements in each sample space.

What are the PDFs for the variables A and RM?

The PDF's for A and RM are

$$F(A) = \frac{1}{12} \text{ and } F(RM) = \frac{1}{12}$$

(b) Find the average monthly production of A.

$$\mu = E(X) = \sum_x x f(x)$$

$$\mu_A = E(A) = \sum_A A\left(\frac{1}{12}\right) = 75 \text{ liters}$$

(c) Find the average monthly production of B. B is just a function of A and RM.

$$\mu_{h(x,y)} = E(h(X,Y)) = \sum_x \sum_y h(x,y) f(x,y)$$

$$\mu_{B(A,RM)} = E(B(A,RM)) = \sum_A \sum_{RM} B(A,RM) f(A)$$

$$\mu_{B(A,RM)} = \sum_{A,RM} \left[RM - \frac{A}{2} + 4\right]\left(\frac{1}{12}\right) = 123.3 \text{ liters}$$

(d) Find the average monthly usage of RM.

$$\mu = E(X) = \sum_x x f(x)$$

$$\mu_{RM} = E(RM) = \sum_{RM} RM\left(\frac{1}{12}\right) = 120.8 \text{ liters}$$

(e) Find the mean of the total monthly production, T.

$E(aX + b) = aE(X) + b$ so

$$E(T) = E(A + B) = E(A) + E(B) = 75 + 123.3333 = 198.3 \text{ liters}$$

(f.1) Find the variance of the monthly production of A using the rigorous definition of the variance.

$$\sigma_A{}^2 = E\left[(A - \mu_A)^2\right] = \sum_A (A - \mu_A)^2 f(A)$$

Month	A	$(A-\mu_A)^2$	$f(A)$	$(A-\mu_A)^2 f(A)$
Jan	50	625	0.0833	52.08333
Feb	50	625	0.0833	52.08333
Mar	60	225	0.0833	18.75
Apr	70	25	0.0833	2.083333
May	80	25	0.0833	2.083333
Jun	90	225	0.0833	18.75
Jul	100	625	0.0833	52.08333
Aug	100	625	0.0833	52.08333
Sep	90	225	0.0833	18.75
Oct	80	25	0.0833	2.083333
Nov	70	25	0.0833	2.083333
Dec	60	225	0.0833	18.75
$\mu_A =$	75	$\sigma_A{}^2 =$		291.67

$$\sigma_A{}^2 = 291.7 \text{ liters squared}$$

(f.2) Find the variance of the monthly production of A using the "mean of the squares minus the square of the mean" formula.

$$\sigma^2 = E\left[X^2\right] - E\left[X\right]^2$$

$$\sigma_A{}^2 = E\left[A^2\right] - E\left[A\right]^2 = 5916.7 - 75^2 = 291.7 \text{ liters squared}$$

This is the same result as was obtained in part (f.1).

(g) Find the variance of the monthly usage of RM.

$$\sigma_{RM}{}^2 = E\left[RM^2\right] - E\left[RM\right]^2 = 14691.7 - 120.8333^2 = 91.0 \text{ liters squared}$$

(h.1) Find the variance of the monthly production of B from tabulated values of B.
Once the values of B, have been tabulated. You can consider it as a random variable itself, rather than as a function of A and RM.

$$\sigma_B{}^2 = E\left[B^2\right] - E\left[B\right]^2 = 15262.5 - 123.3333^2 = 51.4 \text{ liters squared}$$

(h.2) Find the variance of the monthly production of B from the variances of A and RM and the formula for B given in this problem statement.

To determine the variance of the function B(A, RM),

$$B = RM - \frac{A}{2} + 40$$

we need to recall the rules of the variance of a linear combination of RVs:

$$\sigma_{aX+b}^2 = a^2\sigma_X^2 \text{ and } \sigma_{aX+bY}^2 = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\sigma_{XY}$$

Now, the first statement tells us that the variance of the constant 40 is zero. Furthermore, we can see how to handle the variance of the -1/2 in front of A. The second equation tells us that we need to calculate, in addition to the variance of A and RM, the covariance of A, RM. The covariance is given by

$$\sigma_{XY} = E\left[XY\right] - E\left[X\right]E\left[Y\right] = \mu_{XY} - \mu_X\mu_Y$$

we know the average of A and the average of RM, but we need the average of the product of A and RM.

$$\mu_{A\cdot RM} = E(A\cdot RM) = \sum_A A\cdot RM\left(\frac{1}{12}\right) = 9175 \text{ liters squared}$$

Now that we have this mean, we can calculate the covariance:

$$\sigma_{A \cdot RM} = E[A \cdot RM] - E[A]E[RM] = 9175 - 75(120.8333)$$

$$\sigma_{A \cdot RM} = 112.5 \text{ liters squared.}$$

Now, that we have the covariance, we can obtain the variance of the function B:

$$\sigma_B^2 = \sigma_{RM-A/2+40}^2 = \sigma_{RM}^2 + \left(\frac{-1}{2}\right)^2 \sigma_A^2 + 2\left(\frac{-1}{2}\right)\sigma_{A \cdot RM}$$

$$\sigma_{\mathbf{B}}^2 = 90.9722 + \left(\frac{-1}{2}\right)^2 291.6667 + 2\left(\frac{-1}{2}\right)112.5$$

$$\sigma_{\mathbf{B}}^2 = 51.4 \text{ liters squared.}$$

This is the same result as was obtained in part (h.1).

(i) Find the variance of the total monthly production, T.

To determine the variance of the function T(A, B),

$$T(A, B) = B + A$$

we need to recall the rules of the variance of a linear combination of RVs:

$$\sigma_{aX+b}^2 = a^2 \sigma_X^2 \text{ and } \sigma_{aX+bY}^2 = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\sigma_{XY}$$

So

$$\sigma_T^2 = \sigma_{A+B}^2 = \sigma_A^2 + \sigma_B^2 + 2\sigma_{AB}$$

The second equation tells us that we need to calculate, in addition to the variance of A and B, the covariance of A, RM. The covariance of A and B is given by

$$\sigma_{AB} = E[AB] - E[A]E[B] = \mu_{AB} - \mu_A \mu_B$$

we know the average of A and the average of B, but we need the average of the product of A and B.

$$\mu_{A \cdot B} = E(A \cdot B) = \sum_A A \cdot B\left(\frac{1}{12}\right) = 9216.67 \text{ liters squared}$$

Now that we have this mean, we can calculate the covariance:

$$\sigma_{A \cdot B} = E[A \cdot B] - E[A]E[B] = 9216.67 - 75(123.3333)$$

$$\sigma_{A \cdot B} = -33.33 \text{ liters squared.}$$

Now, that we have the covariance, we can obtain the variance of the function T:

$$\sigma_T^2 = \sigma_{A+B}^2 = \sigma_A^2 + \sigma_B^2 + 2\sigma_{A\cdot B}$$

$$\sigma_T^2 = 291.6667 + 51.4 + 2 * (-33.33)$$

$$\sigma_T^2 = 276.4 \text{ liters squared.}$$

(j,k,l,m) Find the standard deviations of A, B, and RM.

$$\sigma_A = \sqrt{\sigma_A{}^2} = \sqrt{291.7} = 17.1 \text{ liters}$$

$$\sigma_B = \sqrt{\sigma_B{}^2} = \sqrt{51.4} = 7.2 \text{ liters}$$

$$\sigma_{RM} = \sqrt{\sigma_{RM}{}^2} = \sqrt{91.0} = 9.5 \text{ liters}$$

$$\sigma_T = \sqrt{\sigma_T{}^2} = \sqrt{276.4} = 16.63 \text{ liters}$$

(n) Find the covariance of A and B.
In part (i), we determined the covariance to be:

$$\sigma_{A\cdot B} = E[A\cdot B] - E[A]E[B] = 9216.67 - 75(123.3333)$$

$$\sigma_{A\cdot B} = -33.33 \text{ liters squared.}$$

(o) Find the correlation coefficient of A and B

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

so

$$\rho_{AB} = \frac{-33.33}{\sqrt{291.7}\cdot\sqrt{51.4}} = -0.272$$

(p) Give a physical description of what the value and sign of the correlation coefficient means.

(1) The negative sign of the correlation coefficient indicates that as A increases, B decreases. This makes sense since A and B are competing for the same raw materials. (We can also see this from the definition of B in the problem statement.)
(2) The value of ρ_{AB} (remember $-1 \le \rho_{AB} \le 1$) indicates that A and B are relatively strongly correlated. We know from the definition of B, that in fact A and B are linearly related.

(extra) The fact that ρ_{AB} is not -1 means that the monthly fluctuation of the raw material, RM, is large enough that disrupts much evidence of the actual linear relationship between A and B. If the Raw Material supply, RM, was a constant, then $\rho_{AB} = -1$.

Problem 2.

A chemical plant contains a jacketed vessel in which the following isomerization reaction takes place:

$$A \rightarrow B$$

The rate of the production of B, r_B [moles/hour], is given by

$$r_B = kC_A$$

where C_A is the concentration of A [moles/liter] and the reaction rate constant, k [liters/hour], is given as a function of the temperature, T [Kelvin], as

$$k = 20.0 \cdot e^{-\frac{10{,}000}{RT}}$$

where R is the gas constant [8.314 J/mole/K]. This (highly ideal) jacketed vessel keeps temperature perfectly constant at the set temperature of 400 K. The concentration in the tank is obtained from the mass balance

$$accumulation = in - out + generation$$

$$V\frac{dC_A}{dt} = QC_{A,in} - QC_A - kC_A$$

where Q is the volumetric flowrate [liters/hour], and has a numerical value of $Q = 9.0$ l/hour. V is the reactor volume, $V = 100.0$ liters. Rearrangement yields:

$$\left(\frac{V}{QC_{A,in} - QC_A - kC_A}\right)dC_A = dt$$

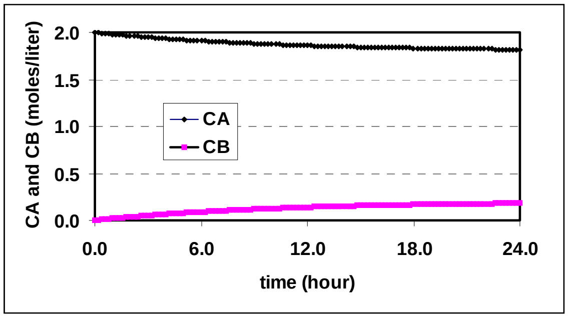
and where $C_{A,in}$ is the inlet concentration of A, $C_{A,in} = 2.0$ mole/liter. We can integrate this equation to yield

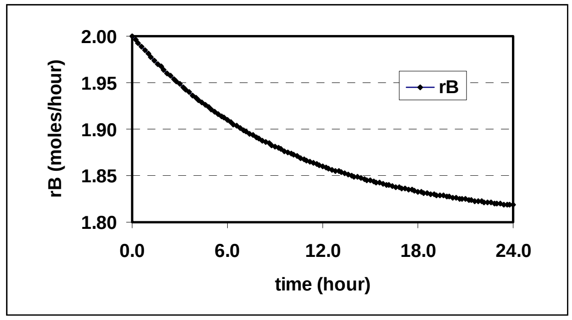
$$\frac{V}{(Q+k)}\ln\left(\frac{kC_{A,in}}{QC_{A,in} - QC_A - kC_A}\right) = t$$

We can rearrange this equation to give us C_A

$$C_A(t) = \frac{C_{A,in}}{(Q+k)}\left(Q + ke^{-\left(\frac{Q+k}{V}\right)t}\right)$$

Plot C_A and C_B on one graph and plot r_B as functions of t for $0 \le t \le 24$ hour. Remember, $C_B = C_{A,in} - C_A$.





We want the average concentration of reactant during that first day of operation. Our formula for the mean over the range a to b is:

$$\mu_{h(x)} = E(h(X)) = \int_a^b h(x) f(x) dx$$

For our problem at hand, identify x, a, b, $h(x)$, and $f(x)$.

Solution:

x is t, a is 0, b is 24, h(x) is $C_A(t)$, $f(x) = \dfrac{1}{b-a} = \dfrac{1}{24}$

(c) What is the average concentration of reactant, C_A, during that first day of operation?

Solution:

$$\mu_{C_A(t)} = \int_0^1 \left(\frac{C_{A,in}}{(Q+k)} \left(Q + ke^{-\left(\frac{Q+k}{V}\right)t} \right) \right) \left(\frac{1}{24-0} \right) dt$$

$$\mu_{C_A(t)} = \left(\frac{C_{A,in}}{24(Q+k)} \left(Qt - \frac{Vke^{-\left(\frac{Q+k}{V}\right)t}}{(Q+k)} \right) \right)_0^{24}$$

$$= \frac{C_{A,in}}{24(Q+k)} \left[\left(24Q - \frac{Vke^{-24\left(\frac{Q+k}{V}\right)}}{(Q+k)} \right) - \left(0 - \frac{Vk}{(Q+k)} \right) \right]$$

$$= \frac{1}{120} \left(216 - 10e^{-2.397312} + 10 \right) = 1.877089 \frac{moles}{liter}$$

(d) What is the average rate of production, r_B, during that first day of operation?

$$r_B = kC_A$$

$$k = 20.0 \cdot e^{-\frac{10,000}{RT}} 20.0 \cdot e^{-\frac{10,000}{8.314 \cdot 400}} = 0.98882$$

In this problem, k, is a constant. Then using the rules of linear operators (and the answer from part (c)):

$$E[r_B] = E[kC_A] = kE[C_A] = (0.98882)(1.877089) = 1.856101 \frac{moles}{hour}$$

(e) What is the average concentration of B, C_B, during that first day of operation?

$$C_B = C_{A,in} - C_A$$

In this problem, $C_{A,in}$, is a constant. Then using the rules of linear operators (and the answer from part (c)):

$$E[C_B] = E[C_{A,in} - C_A] = E[C_{A,in}] - E[C_A]$$

$$E[C_B] = C_{A,in} - E[C_A] = 2 - 1.877089 = 0.122911 \frac{moles}{liter}$$

(f) What is the variance of C_A during that first day of operation?

Solution:
We know the variance of a function is given by:

$$\sigma_{g(x)}^2 = E\left[g(X)^2\right] - E[g(X)]^2$$

We have already calculated the second term on the right hand side in part (c). We must calculate the first term on the right hand side.

$$E\left[g(X)^2\right] = \int_0^1 \left(\frac{C_{A,in}}{(Q+k)}\left(Q + ke^{-\left(\frac{Q+k}{V}\right)t}\right)\right)^2 \left(\frac{1}{24-0}\right)dt$$

$$E\left[g(X)^2\right] = \frac{1}{24}\left(\frac{C_{A,in}}{(Q+k)}\right)^2 \int_0^1 \left(Q^2 + 2Qke^{-\left(\frac{Q+k}{V}\right)t} + ke^{-2\left(\frac{Q+k}{V}\right)t}\right)dt$$

$$E\left[g(X)^2\right] = \frac{1}{24}\left(\frac{C_{A,in}}{(Q+k)}\right)^2 \left(Q^2 t - \frac{2QVke^{-\left(\frac{Q+k}{V}\right)t}}{(Q+k)} - \frac{Vk^2 e^{-2\left(\frac{Q+k}{V}\right)t}}{2(Q+k)}\right)_{t=0}^{t=24}$$

$$E\left[g(X)^2\right] = 3.525934056 \frac{moles^2}{liter^2}$$

$$\sigma_{C_A}^2 = E\left[{C_A}^2\right] - E[C_A]^2 = 3.525934056 - 1.877089^2 = 0.00247 \frac{moles^2}{liter^2}$$

(g) What is the variance of r_B during that first day of operation?

$$r_B = kC_A$$

In this problem, k, is a constant. Then using the rules of linear operators (and the answer from part (f)):

$$\sigma_{r_B}^2 = E\left[(kC_A)^2\right] - E[kC_A]^2 = k^2 E\left[{C_A}^2\right] - k^2 E[C_A]^2 = k^2 \sigma_{C_A}^2$$

$$\sigma_{r_B}^2 = k^2 \sigma_{C_A}^2 = 0.98882^2 0.00247 = 0.00242 \frac{moles^2}{hour^2}$$

(h) What is the variance of C_B during that first day of operation?

Solution:

$$C_B = C_{A,in} - C_A$$

Remember, the rule for linear operators:

$$\sigma_{aX+bY}^2 = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\sigma_{XY}$$

so

$$\sigma_{C_B}^2 = \sigma_{C_{A,in}-C_A}^2 = \sigma_{C_{A,in}}^2 + (-1)^2\sigma_{C_A}^2 - 2\sigma_{C_{A,in}C_A}$$

Now $C_{A,in}$ is a constant so its variance and covariance are zero.

$$\sigma_{C_B}^2 = 0 + (-1)^2\sigma_{C_A}^2 + 0 = \sigma_{C_A}^2 = 0.00247\frac{moles^2}{liter^2}$$

Problem 3.

A private pilot wishes to insure his airplane for $200,000. The insurance company estimates that a total loss may occur with a probability of 0.002, a 50% loss with probability 0.01 and a 25% loss with a probability of 0.1. Ignoring all other partial losses, what premium should the insurance company charge each year to realize an average profit of $500?

solution:

The information given:

- Insure the plane for $200,000.
- P(total loss)=0.002
- P(half loss)=0.01
- P(quarter loss)=0.1
- P(no loss)=remainder=1-0.1-0.01=0.002=0.888.

What premium should an insurance company charge for an average annual profit of $500?

Find average loss per year.

$$\mu = E(h(x)) = \sum_x h(x) f(x)$$

Let h(x) be the amount paid.

$$\mu = \$0 \cdot (0.88) + \$200{,}000 \cdot (0.002) + \$100{,}000 \cdot (0.01) + \$50{,}000 \cdot (0.1)$$
$$\mu = \$6400$$

Therefore the annual premium should be $500 dollars more than the mean or: $6900.