Homework Assignment Number Two Solutions

Problem 1.
Determine the value of c so that the following functions can serve as a PDF of the discrete random variable *X*.

$f(x) = c\left(x^2 + 4\right)$ where $x = 0,1,2,3;$

Solution:
(a) $f(x) = c(x^2 + 4)$ for $x = 0,1,2,3;$

In order for this to be a valid PDF, we require that

$$\sum_x f(x) = 1$$

In this case,

$$\sum_x f(x) = 1 = c(0^2 + 4) + c(1^2 + 4) + c(2^2 + 4) + c(3^2 + 4)$$
$$\sum_x f(x) = 1 = c(4 + 5 + 9 + 13)$$

Solving for *c* yields,

$$c = \frac{1}{30}$$

Problem 2.
A shipment of 7 computer monitors contains 2 defective monitors. A business makes a random purchase of 3 monitors. If x is the number of defective monitors purchased by the company, find the probability distribution of X. (This means you need three numbers, f(x=0), f(x=1), and f(x=2) because the random variable, X = number of defective monitors purchased, has a range from 0 to 2. Also, find the cumulative PDF, F(x). Plot the PDF and the cumulative PDF. These two plots must be turned into class on the day the homework is due.

Solution:
We have 7 TVs, 2 defects, 3 purchases. Therefore we have 5 good TV (G) and 2 defects (D). X is the number of D purchased. X can take values of 0, 1, and 2 defects.

The total number of ways to take 3 TVs from a set of 7 TVs is:

$$\binom{7}{3} = \frac{7!}{3!(7-3)!} = \frac{7*6*5*4!}{6*4!} = 35$$

We use the combination rule because order does not matter.

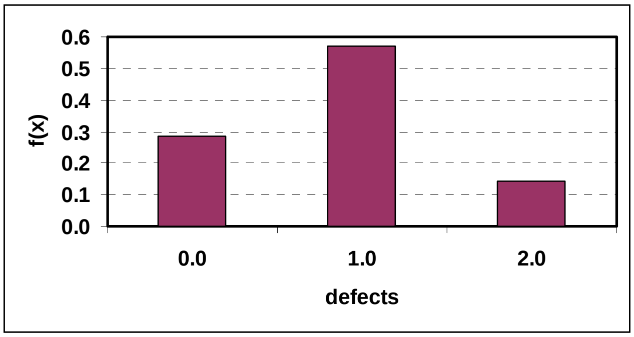
We need to determine of these 35 ways to select TVs, the number of ways we can get 0, 1, and 2 defects.

The number of ways of taking x of the 2 defective TV's from the set of 7 is given by the number of ways we can take x from 2 and (3-x) from 5.

$$\binom{2}{x}\binom{5}{3-x} = \frac{2!}{(2-x)!x!}\frac{5!}{[5-(3-x)]!(3-x)!} = \frac{2!}{(2-x)!x!}\frac{5!}{(2+x)!(3-x)!}$$

When x is 0, this yields 10
When x is 1, this yields 20
When x is 2, this yields 5

The probability P(X=x) is then the number of ways to get x defects from 35 total ways.
P(X=0) = 10/35 = 2/7
P(X=1) = 20/35 = 4/7
P(X=2) = 5/35 = 1/7
This makes sense, because the sum of all our probabilities are 1.0.
The graphical histogram looks like:



The cumulative PDF is plotted below.

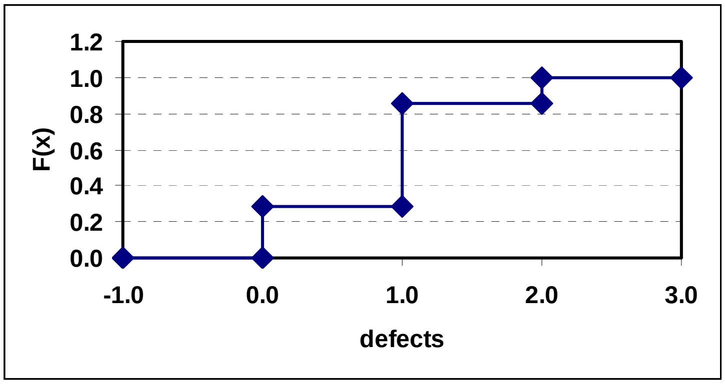
For a discrete random variable, the cumulative distribution is given by

$$F(x) = P(X \le x) = \sum_{t \le x} f(t) \quad \text{for} \quad \text{all allowable values of } x$$

$$F(x=0) = P(X \le 0) = \sum_{t \le 0} f(t) = \frac{2}{7}$$

$$F(x=1) = P(X \le 1) = \sum_{t \le 1} f(t) = \frac{2}{7} + \frac{4}{7} = \frac{6}{7}$$

$$F(x=2) = P(X \le 2) = \sum_{t \le 2} f(t) = \frac{2}{7} + \frac{4}{7} + \frac{1}{7} = \frac{7}{7} = 1$$



Problem 3.
A continuous random variable, X, that can assume values between x=2 and x=5 has a PDF given by

$$f(x) = \frac{2}{27}(1+x)$$

Find (a) P(X<4) and find (b) P(3<X<4). Plot the PDF and the cumulative PDF. These two plots must be turned into class on the day the homework is due.

solution:

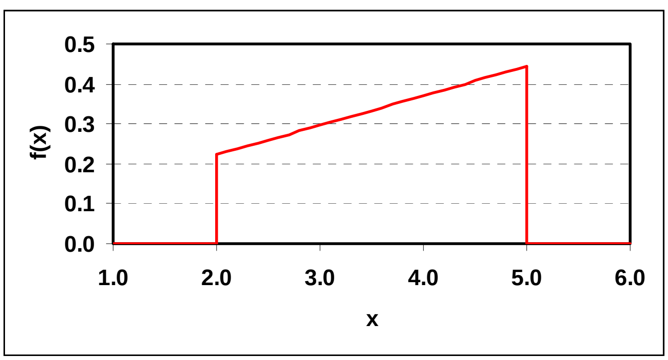
(a) $$P(X<4) = P(2<X<4) = \int_2^4 f(x)dx$$

$$P(X<4) = \int_2^4 f(x)dx = \left[\frac{2x}{27} + \frac{x^2}{27}\right]_{x=2}^{x=4} = \frac{24}{27} - \frac{8}{27} = \frac{16}{27}$$

(b) $$P(3<X<4) = \int_3^4 f(x)dx$$

$$P(3<X<4) = \int_3^4 f(x)dx = \left[\frac{2x}{27} + \frac{x^2}{27}\right]_{x=3}^{x=4} = \frac{24}{27} - \frac{15}{27} = \frac{9}{27} = \frac{1}{3}$$

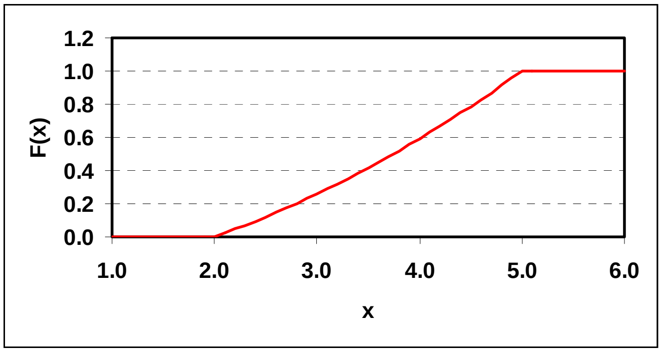
Here is a plot of the PDF:



The cumulative PDF is given as the probability that x

$$F(x) = P(X < x) = \int_2^x f(x')dx' = \left[\frac{2x'}{27} + \frac{x'^2}{27}\right]_{x'=2}^{x'=x} = \frac{2x}{27} + \frac{x^2}{27} - \frac{8}{27}$$

Here is a plot of the cumulative PDF:



Problem 4.
Consider a system of particles that sit in an electric field where the energy of interaction with the electric field is given by E(x) = 2477.572 + 4955.144x, where x is spatial position of the particles. The probability distribution of the particles is given by statistical mechanics to be f(x) = c*exp(-E(x)/(R*T)) for 0<x<1 and 0 otherwise, where R = 8.314 J/mol/K and T = 270.0 Kelvin.

(a) Find the value of c that makes this a legitimate PDF.
(b) Find the probability that a particles sits at x<0.25
(c) Find the probability that a particles sits at x>0.75
(d) Find the probability that a particles sits at 0.25<x<0.75

Solution:
(a)

$$\int_0^1 f(x)dx = \int_0^1 ce^{-\frac{E(x)}{RT}}dx = c\int_0^1 e^{-\frac{2477.572+4955.144x}{8.314*270}}dx = 1$$

$$c = \frac{1}{\int_0^1 e^{-\frac{2477.572+4955.144x}{8.314*270}}dx} = \frac{1}{\left[-\frac{8.314*270}{4955.144}e^{-\frac{2477.572+4955.144x}{8.314*270}}\right]_0^1} = 7.478557$$

(b)

$$P(x < 0.25) = \int_0^{0.25} f(x)dx = \int_0^{0.25} ce^{-\frac{E(x)}{RT}}dx = 0.476529$$

(c)

$$P(x > 0.75) = \int_{0.75}^1 f(x)dx = \int_{0.75}^1 ce^{-\frac{E(x)}{RT}}dx = 0.091010$$

(d)

$$P(0.25 < x < 0.75) = \int_{0.25}^{0.75} f(x)dx = \int_{0.25}^{0.75} ce^{-\frac{E(x)}{RT}} dx = 0.432461$$

We see that (b) + (c) + (d) = 1, as they must. We also see that more particles sit in the first quarter of the range from 0 to 0.25 than in the last quarter from 0.75 to 1.0. Why is that? Because statistical mechanics tell us that particles prefer to rest in lower energies. The energy, E(x), is lower in the first quarter than in the last quarter. Plot it to see for yourself.

Problem 5.

Let X denote the reaction time, in seconds, to a certain stimulant and Y denote the temperature (reduced units) at which a certain reaction starts to take place. Suppose that the random variables X and Y have the joint PDF,

$$f(x,y) = \begin{cases} cxy \text{ for } 0 < \text{x} < 1; 0 < \text{y} < 2.1 \\ 0 \quad \text{elsewhere} \end{cases}$$

where c = 0.907029.

Find (a) $P\left(0 \le X \le 1/2 \text{ and } 1/4 \le Y \le 1/2\right)$ and (b) $P(X < Y)$.

Solution:

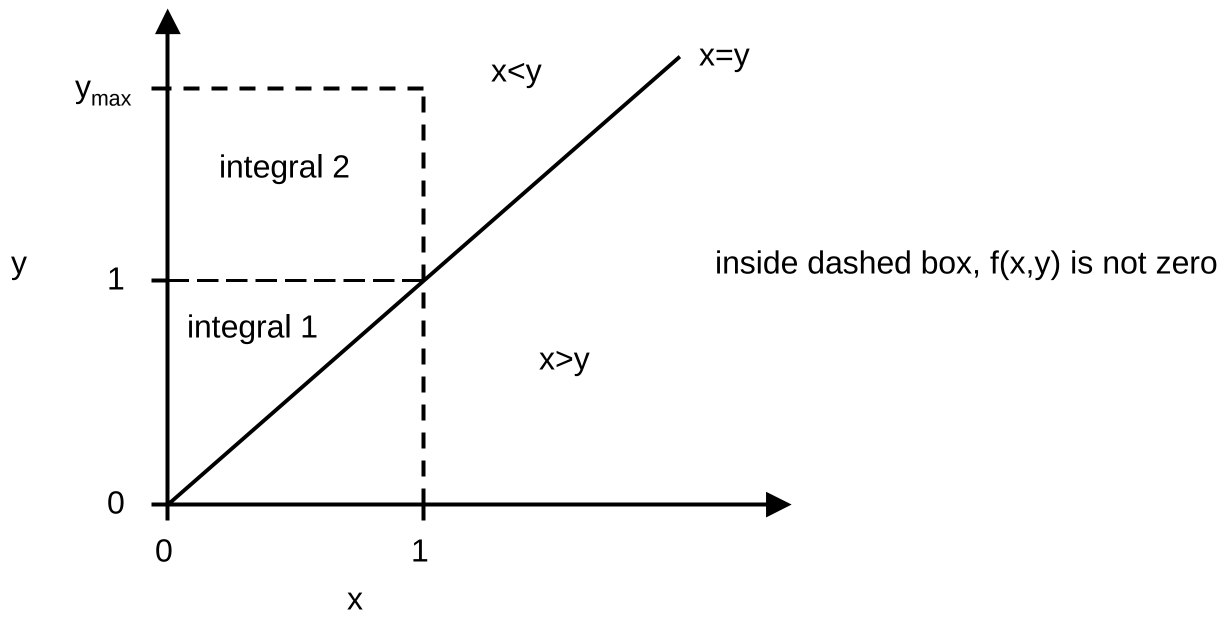
(a) Find $P\left(0 \le X \le \frac{1}{2} \; and \; \frac{1}{4} \le Y \le \frac{1}{2}\right)$

$$P\left(0 \le X \le \frac{1}{2} \; and \; \frac{1}{4} \le Y \le \frac{1}{2}\right) = \int_{1/4}^{1/2}\int_{0}^{1/2} f(x,y)dxdy = \int_{1/4}^{1/2}\int_{0}^{1/2} cxydxdy$$

$$P\left(0 \le X \le \frac{1}{2} \; and \; \frac{1}{4} \le Y \le \frac{1}{2}\right) = c\int_{1/4}^{1/2} y\left(\frac{x^2}{2}\right)\Bigg|_0^{1/2} dy = \frac{c}{8}\int_{1/4}^{1/2} ydy = \frac{c}{8}\int_{1/4}^{1/2} ydy = \frac{3c}{256} = 0.01063$$

(b) Find $P(X < Y)$

Here we are integrating over an odd shape of the x-y plane. We want x to be less than y but x must also be less than its maximum value as specified by the PDF, in this case 1. The easiest way to do this integration is to split the integral into 2 parts, then add them together.



$$\text{integral\_1} = \int_0^1\int_0^y f(x,y)dxdy = \int_0^1\int_0^y cxydxdy = c\int_0^1 y\left(\frac{x^2}{2}\right)\Bigg|_0^y dy = \frac{c}{2}\int_0^1 y^3 dy = \frac{c}{8}y^4\Big|_0^1 = \frac{c}{8}$$

$$\text{integral\_2} = \int_1^{y_{max}}\int_0^1 f(x,y)dxdy = \int_1^{y_{max}}\int_0^1 cxydxdy = c\int_1^{y_{max}} y\left(\frac{x^2}{2}\right)\Bigg|_0^1 dy = \frac{c}{2}\int_1^{y_{max}} ydy = \frac{c}{4}y^2\Big|_1^{y_{max}} = \frac{c}{4}\left(y_{max}^2 - 1\right)$$

$$P(X<Y) = \text{integral\_1} + \text{integral\_2} = \frac{c}{8} + \frac{c}{4}\left(y_{max}^2 - 1\right) = \frac{c}{8}\left(2y_{max}^2 - 1\right) = 0.8866$$

Problem 6.

Let X denote the number of times that a control machine malfunctions per day (choices: 1, 2, 3) and Y denote the number of times a technician is called. f(x,y) is given in tabular form.

f(x,y)	**x**	**1**	**2**	**3**
y	1	0.05	0.05	0.1
	2	0.05	0.1	0.35
	3	0.0	0.2	0.1

(a) Evaluate the marginal distribution of X.
(b) Evaluate the marginal distribution of Y.
(c) Find P(Y = 3|X = 2).
(d)

Solution:

(a) Evaluate the marginal distribution of X

$$g(x) = \sum_y f(x,y)$$

$$g(x=1) = \sum_y f(x=1,y) = f(1,1) + f(1,2) + f(1,3) = 0.05 + 0.05 + 0.0 = 0.1$$

$$g(x=2)=\sum_{y} f(x=2,y)= f(2,1)+ f(2,2)+ f(2,3)=0.05+0.1+0.2=0.35$$
$$g(x=3)=\sum_{y} f(x=3,y)= f(3,1)+ f(3,2)+ f(3,3)=0.1+0.35+0.1=0.55$$

(b) Evaluate the marginal distribution of Y

$$h(y)=\sum_{x} f(x,y)$$
$$h(y=1)=\sum_{x} f(x,y=1)= f(1,1)+ f(2,1)+ f(3,1)=0.05+0.05+0.1=0.2$$
$$h(y=2)=\sum_{x} f(x,y=2)= f(1,2)+ f(2,2)+ f(3,2)=0.05+0.1+0.35=0.5$$
$$h(y=3)=\sum_{x} f(x,y=3)= f(1,3)+ f(2,3)+ f(3,3)=0.0+0.2+0.1=0.3$$

(c) Find $P(Y=3\,|\,X=2)$

$$P(Y=3\,|\,X=2)= f(y=3|x=2)=\frac{f(x=2,y=3)}{g(x=2)}=\frac{0.2}{0.35}=\frac{4}{7}$$