Homework Assignment Number One Solutions

Problem 1. Permutations & Combinations

Consider sputtering an film with 8 layers that are designated A-rich, A-lean, B-rich, B-lean, C-rich, C-lean, D-rich and D-lean. In how many different ways can the layers be put down
(a) with no restrictions?
(b) if the corresponding rich and lean layers must be adjacent?
(c) if all the rich layers are sputtered first, followed by all the lean layers?

Solution:

Since we are concerned with the arrangement, you must consider that order matters, and thus use the formula for permutations, not combinations.

(a) Arranging 8 layers in any way is given by the permutation formula with n=8 and r=8

$$_{n}P_{r} = \frac{n!}{(n-r)!}$$

so

$$_{8}P_{8} = \frac{8!}{(8-8)!} = \frac{8!}{0!} = 8! = 40{,}320$$

(b) If each rich and lean layer is adjacent, then you only have four objects.

$$_{4}P_{4} = \frac{4!}{(4-4)!} = \frac{4!}{0!} = 4! = 24$$

However each of those four objects has two possible choices, (rich on bottom and lean on top, or vice versa). So you must multiply this by 2*2*2*2=16. 24*16=**384** ways.

(c) If all the rich layers are sputtered first, followed by all the lean layers, then there are $_{4}P_{4} = 24$ arrangements of rich layers and $_{4}P_{4} = 24$ arrangement of lean layers so the total number of arrangements is 24*24 = 576 arrangements (by the general multiplicative rule).

Problem 2. Permutations & Combinations

(a) A preliminary engineering design involves three stages where different solvents are used to perform liquid-liquid extraction. If you are considering 5 different solvents, each of whichwill be used in no more than one extractor, how many different possible designs would you need to investigate?
(b) Now make the assumption that it doesn't matter what order the three extractions are performed in. How many different possible designs would you need to investigate?

Solution:

(a) This is a question of permutations. The objects are distinct because each stage uses a different solvent and the stages are distinct. The first stage is different than the second stage, etc. The number of ways that 5 solvents can be assigned to 3 extractors (where order matters) is

$$_{n}P_{r} = \frac{n!}{(n-r)!}$$

$$_{5}P_{3} = \frac{5!}{(5-3)!} = 60$$

(b) This is a question of combinations, because now order doesn't matter.

$$\binom{5}{3} = \frac{5!}{3!(5-3)!} = 10$$

Problem 3. Counting
A drug for the relief of arthritis can be purchased from 8 different manufacturers in liquid, tablet, or capsule form, all of which come in 100, 200 and 400 mg doses. How many different ways can a doctor prescribe the drug to a patient suffering from arthritis?

Solution:
By the multiplicative rule, there are

$$\text{number of ways} = \text{manufactures} \cdot \text{form} \cdot \text{dosage}$$
$$\text{number of ways} = 8 \cdot 3 \cdot 3 = 72$$

Problem 4. Probabilities – Union, Intersection and Sums
A group of 500 college freshman are surveyed on the subject of fossil energy, climate change and sustainability. 210 believe sustainability is a legitimate global issue. 258 believe that anthropogenic climate change exists. 216 believe that fossil energy will be exhausted in their lifetime. 122 believe in both sustainability and anthropogenic climate change. 83 believe in both the exhaustion of fossil energy and anthropogenic climate change. 97 believe in both the sustainability and the exhaustion of fossil energy. 52 believe all three statements.

(a) Find the probability of those who believe in sustainability but not in anthropogenic climate change.
(b) Find the probability of those who believe fossil fuel will be exhausted in their lifetimes and in anthropogenic climate change but not in sustainability.
(c) Find the probability of those who believe in neither sustainability nor that fossil fuel will be exhausted.

Solution:
210/500 believe in sustainability (S)
258/500 believe in anthropogenic climate change (C)
216/500 believe in the exhaustion of fossil energy (E)
122 believe S & C,
83 believe E & C,
97 believe S & E,
52 believe S, E & C.
We write this as

$$P(S) = 0.42 \qquad P(C) = 0.516 \qquad P(E) = 0.432$$
$$P(S \cap C) = 0.244 \qquad P(E \cap C) = 0.166 \qquad P(S \cap E) = 0.194 \qquad P(S \cap E \cap C) = 0.104$$

We can obtain other probabilities through the union and intersection rules:

(a) Find the probability of those who believe in sustainability but not in anthropogenic climate change.

$$P(S) = P(S \cap C') + P(S \cap C)$$

Rearranging for the unknown

$$P(S \cap C') = P(S) - P(S \cap C) = 0.42 - 0.244 = 0.176$$

(b) Find the probability of those who believe fossil fuel will be exhausted in their lifetimes and in anthropogenic climate change but not in sustainability.

$$P(E \cap C) = P(E \cap C \cap S') + P(E \cap C \cap S)$$

Rearranging for the unknown

$$P(E \cap C \cap S') = P(E \cap C) - P(E \cap C \cap S) = 0.166 - 0.104 = 0.062$$

(c) Find the probability of those who believe in neither sustainability nor that fossil fuel will be exhausted.

To find the students who are not E or S, we first find the students who are E, S or C, using the union rule for three groups:

$$P(E \cup D \cup S) = P(S) + P(E) + P(C) - P(E \cap C) - P(S \cap C) - P(E \cap S) + P(E \cap C \cap S) = 0.868$$

Students are either E, S or C or do none of them so:

$$1.0 = P(E \cup C \cup S) + P(E' \cap C' \cap S')$$

Rearranging:

$$P(E' \cap C' \cap S') = 1.0 - P(E \cup C \cup S) = 1.0 - 0.868 = 0.132$$

Now in order to find those that are not E nor S, we can divide them into 2 groups those that are C and those are not C:

$$P(E' \cap S') = P(E' \cap S' \cap C') + P(E' \cap S' \cap C)$$

We have already obtained the first term on the right hand side of the equation. We can obtain the second term on the RHS by realizing that people who are S and C either are E or not E:

$$P(S \cap C) = P(S \cap C \cap E') + P(S \cap C \cap E)$$

Rearranging

$$P(S \cap C \cap E') = P(S \cap C) - P(S \cap C \cap E) = 0.244 - 0.104 = 0.140$$

Then people who are C are divided into four groups based on their S and E status (see the Venn Diagram)

$$P(C) = P(S' \cap C \cap E) + P(S' \cap C \cap E') + P(S \cap C \cap E') + P(S \cap C \cap E)$$

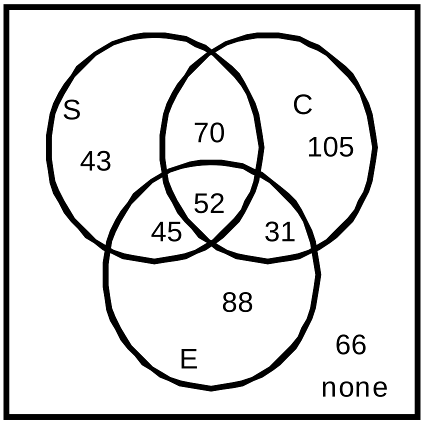
Rearrange this:

$$P(S' \cap C \cap E') = P(C) - P(S' \cap C \cap E) - P(S \cap C \cap E') - P(S \cap C \cap E)$$
$$= 0.516 - 0.062 - 0.140 - 0.104 = 0.210$$

Plug this number back in for the second term on the RHS of the equation for $P(E' \cap S')$

$$P(E' \cap S') = 0.132 + 0.210 = 0.342$$

Using these numbers we can complete a Venn Diagram, if we wanted.



total population = 500 students

Problem 5. Conditional Probability
In sampling a population for the presence of a disease, the population is of two types: Infected and Uninfected. The results of the test are of two types: Positive and Negative. In rare disease detection, a high probability for detecting a disease can still lead to more false positives than true positives. Consider a case where a disease infects 1 out of every 100,000 individuals. The probability for a positive test result given that the subject is infected is 0.99. The probability for a negative test result given that the subject is uninfected is 0.999.
(a) What is the probability of being uninfected?
(b) What is the probability of being uninfected AND testing negative?
(c) What is the probability of being uninfected AND testing positive?
(d) What is the probability of testing positive?
(e) What is the probability of being uninfected given that the person tested positive? (This is the percentage of erroneous positive tests.)
(Make sure your answers are accurate to five significant figures.)

Solution:

We are told:

$$P(I) = 10^{-5}$$

$$P(N|U) = \frac{P(N \cap U)}{P(U)} = 0.999$$

$$P(P|I) = \frac{P(P \cap I)}{P(I)} = 0.99$$

For testing a single person, define the complete sample space.

The sample space is $S = \{IP, IN, UP, UN\}$ where I = Infected, U=Uninfected, P=positive test result, N=negative test result. The Venn Diagram looks like this:

Infected $\cap$ Positive	Infected $\cap$ Negative
Uninfected $\cap$ Positive	UnInfected $\cap$ Negative

When you have a simple sample space like this, you can see 8 additional constraints on the system, in addition to the union, conditional, and intersection rules.

$$P(P) = P(P \cap I) + P(P \cap U) \qquad P(I \mid P) + P(U \mid P) = 1$$

$$P(N) = P(N \cap I) + P(N \cap U) \qquad P(I \mid N) + P(U \mid N) = 1$$
$$P(U) = P(U \cap P) + P(U \cap N) \qquad P(P \mid U) + P(N \mid U) = 1$$
$$P(I) = P(I \cap P) + P(I \cap N) \qquad P(P \mid I) + P(N \mid I) = 1$$

We can solve this problem by finding a variety of probabilities:

$$P(U) + P(I) = 1 \quad \text{so} \quad P(U) = 1 - P(I) = 0.99999$$

$$P(N \cap U) = P(N \mid U)P(U) = 0.999 \cdot 0.99999 = 0.99899001$$
$$P(P \cap I) = P(P \mid I)P(I) = 0.99 \cdot 0.00001 = 0.0000099$$

$$P(U) = P(U \cap P) + P(U \cap N) \quad \text{so} \quad P(P \cap U) = P(U) - P(N \cap U) = 0.00099999$$
$$P(I) = P(I \cap P) + P(I \cap N) \quad \text{so} \quad P(N \cap I) = P(I) - P(P \cap I) = 1.0E-7$$

$$P(P) = P(P \cap I) + P(P \cap U) = 0.00100989$$
$$P(N) = P(N \cap I) + P(N \cap U) = 0.99899011$$

$$P(P \mid U) = P(P \cap U)/P(U) = 0.001$$
$$P(N \mid I) = P(N \cap I)/P(I) = 0.01$$
$$P(U \mid N) = P(U \cap N)/P(N) = 0.999999899$$
$$P(U \mid P) = P(U \cap P)/P(P) = 0.990196952$$
$$P(I \mid N) = P(I \cap N)/P(N) = 0.000001001$$
$$P(I \mid P) = P(I \cap P)/P(P) = 0.009803047$$

(a) What is the probability of being uninfected?

$$P(U) = 1 - P(I) = 0.99999$$

(b) What is the probability of being uninfected AND testing negative?

$$P(N \cap U) = P(N \mid U)P(U) = 0.999 \cdot 0.99999 = 0.99899001$$

(c) What is the probability of being uninfected AND testing positive?

$$P(P \cap U) = P(U) - P(N \cap U) = 0.00099999$$

(d) What is the probability of testing positive?

$$P(P) = P(P \cap I) + P(P \cap U) = 0.00100989$$

(e) What is the probability of being uninfected given that the person tested positive? (This is the percentage of erroneous positive tests.)

$$P(U \mid P) = P(U \cap P)/P(P) = 0.990196952$$

From part (e) we see the problem of rare disease detection. Even with our accurate method, over 99% of the positive results are actually uninfected.