Applied Statistics and Numerical Methods for Engineers
ChE 301, Fall 1998
Midterm Exam Number One
Administered: Monday, September 21, 1998

In all relevant problems: WRITE DOWN THE FORMULA YOU USE, BEFORE YOU USE IT.
In all problems: NO NUMERICAL INTEGRATION REQUIRED.
ALL PROBLEMS ARE WORTH 2 POINTS. THE EXAM HAS 36 POINTS.

For problems (1.1) to (1.7), consider that you have a group of 6 engineers, named Ali, Barbara, Charlie, Denise, Ernie, and Feng. Ali and Barbara are environmental engineers. Charlie and Denise are material scientists. Ernie and Feng are chemical engineers. They are to be randomly split up into 2 groups of three people.

(1.1) Define the sample space. (Either with words or lists.) Define one example of an event from that sample space.

Solution: The sample space is all possible divisions of 6 people in 2 groups of three. One example of an event in that sample space is: Group 1 contains Ali, Barbara, Charlie and Group 2 contains Denise, Ernie, and Feng. Another example of an event is all groupings in which Ali and Barbara are in the same group.

(1.2) What are the number of ways that 2 groups of three can be formed, irrespective of engineering discipline?

Solution: Since the order of individuals in the group doesn't matter, we must use the combination rule.

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

In this case, n=6. We want the number of combinations of three people. The remaining three will form the second group.

$$\binom{6}{3} = \frac{6!}{3!(6-3)!} = \frac{6*5*4*3*2}{3*2*(3*2)} = 20$$

(1.3) Consider the number of ways that 2 groups can be formed, considering the 6 people only by their profession?

Possible outcomes of a group	Number of ways for each outcome
2 Environmental, 1 Chemical, 0 Material	2
2 Environmental, 0 Chemical, 1 Materials	2
1 Environmental, 2 Chemical, 0 Materials	2
0 Environmental, 2 Chemical, 1 Materials	2
0 Environmental, 1 Chemical, 2 Materials	2
1 Environmental, 0 Chemical, 2 Materials	2
1 Environmental, 1 Chemical, 1 Materials	X

Find X.

Solution: In this case, order doesn't matter. Use the combination rule for choosing from types.

$$X = \text{ways of } (1E, 1C, 1M) = \binom{2}{1}\binom{2}{1}\binom{2}{1} = (2)(2)(2) = 8$$

(1.4) What is the probability of having the 2 chemical engineers in different groups?

Solution: From problem (3), we know that there are 3 outcomes that split the chemical engineers (2E,1C,0M), (0E,1C,2M) and (1E,1C,1M). The number of ways to obtain each of these is:

$$(2E, 1C, 0M) = 2 \text{ and } (0E, 1C, 2M) = 2 \text{ and } (1E, 1C, 1M) = 8$$

The total number of ways to split the chemical engineers is then 12.
Since the total number of ways of splitting is 20,
The probability is thus

$$P(\text{split chem Es}) = \frac{\text{\# of ways to split Chem Es}}{\text{total \# of ways of splits}} = \frac{12}{20} = 0.6$$

(1.5) What is the probability of the 2 chemical engineers in different groups OR the 2 materials scientists in the same group?

Solution: From problem (3), we know that there are 4 outcomes that put the mechanical engineers in the same group (0E,1C,2M) and (1E,0C, 2M) and (2E,1C,0M) and (1E,2C, 0M). The number of ways to obtain each of these is:

$$(0E, 1C, 2M) = 2 \text{ and } (1E, 0C, 2M) = 2$$
$$(2E, 1C, 0M) = 2 \text{ and } (1E, 2C, 0M) = 2$$

The total number of ways to keep the materials scientists together is then 8.
Since the total number of ways of splitting is 20,
The probability is thus

$$P(\text{keep Mech Es together}) = \frac{\text{\# of ways to keep Mech Es}}{\text{total \# of ways of splits}} = \frac{8}{20} = 0.4$$

We are asked for
let A be the event that we split the Chem E's and B be the event that we keep the Mech E's together. We are asked for

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

The intersection of A and B is

$$(0E, 1C, 2M) = 2 \text{ and } (2E, 1C, 0M) = 2$$

so

$$P(A \cap B) = \frac{4}{20} = 0.2$$

thus

$$P(A \cup B) = 0.6 + 0.4 - 0.2 = 0.8$$

(1.6) What is the probability of having the 2 materials scientists in the same group GIVEN the 2 chemical engineers are in different groups.
Solution:

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{0.2}{0.6} = 0.333$$

(1.7) Determine rigorously whether having the 2 materials scientists in the same group and having the 2 chemical engineers in different groups are independent.
Solution: If A and B are independent:

$$P(B \mid A) = P(B)$$

Since $P(B \mid A) = 0.333$ and $P(B) = 0.4$, **A and B are not independent.**

(2.1) State the three conditions that must be satisfied in order for a continuous probability density to be valid.

Solution:

$$f(x) \geq 0 \quad \text{for all } x \in R$$

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$P(a < X < b) = \int_a^b f(x)dx$$

(2.2) For the PDF $f(x) = \begin{cases} \frac{x}{16} & \text{for } 2 \leq x \leq 6 \\ 0 & \text{otherwise} \end{cases}$, find $P(X \geq 4)$.

Solution:

$$P(X \geq 4) = P(4 < X \leq 6) = \int_4^6 f(x)dx = \int_4^6 \frac{x}{16}dx = \frac{20}{32} = \frac{5}{8}$$

(2.3) For the PDF $f(x) = \begin{cases} 0.125 & \text{for } x = 2 \\ 0.375 & \text{for } x = 4 \\ 0.25 & \text{for } x = 6 \\ 0.25 & \text{for } x = 8 \end{cases}$, find $P(2 < X \leq 6)$

Solution:

$$P(2 < X \leq 6) = P(x = 4) + P(x = 6) = 0.375 + 0.25 = 0.625$$

(2.4) Which of the following statements are TRUE?

(a) A histogram is a plot of the cumulative PDF.
(b) The cumulative PDF monotonically increases.
(c) A symmetric PDF gives rise to a symmetric cumulative PDF.
(d) The continuous cumulative PDF, $F(x)$, has a value of 0.5 when $x = \mu_x$.

Solution:

(a) False. A histogram is a plot of the cumulative PDF.
(b) True.

(c) False. A symmetric PDF (about the y axis) does not give rise to a symmetric cumulative PDF. (about the y-axis).

(d) Can be true. Not necessarily true.

For Problems (3.1) to (3.6) Consider the following: x and y are random variables. d and p are functions of x and y,
d = y - x and p = xy. 100 random samples of x and y values are taken. The following data is then calculated for those 100 samples. (Not all of the raw data is shown below.)

run	x	y	d	p	x^2	y^2	d^2	p^2	xd	xy	yp
1	10.90	18.74	7.85	204.30	118.79	351.37	61.55	41739.44	85.51	186.69	3829.59
2	11.17	19.73	8.55	220.36	124.80	389.08	73.17	48556.36	95.56	172.26	4346.53
3	9.34	20.14	10.80	188.14	87.23	405.77	116.72	35395.82	100.91	239.70	3789.78
…	…	…	…	…	…	…	…	…	…	…	…
98	11.32	18.41	7.09	208.43	128.20	338.87	50.21	43443.02	80.23	176.79	3836.85
99	10.57	21.80	11.23	230.47	111.79	475.18	126.01	53118.67	118.69	235.07	5024.02
100	11.69	19.66	7.97	229.70	136.56	386.37	63.53	52763.36	93.14	211.16	4515.11
sum	1013.00	1979.94	XXXX	20041.70	10381.3	39327.6	XXXX	4070754.6	9660.4	19897.98	397789.3
mean	10.13	19.80	XXXX	200.42	103.81	393.28	XXXX	40707.55	96.60	198.98	3977.89

(3.1) Find the variance of x, σ_x^2.

Solution:

$$\sigma_x^2 = E\left[X^2\right] - E[X]^2 = 103.81 - 10.13^2 = 1.19$$

(3.2) Find the standard deviation of y, σ_y.

Solution:

$$\sigma_y^2 = E\left[Y^2\right] - E[Y]^2 = 393.28 - 19.80^2 = 1.24$$

$$\sigma_y = \sqrt{\sigma_y^2} = \sqrt{1.24} = 1.11$$

(3.3 and 3.4) Find the mean, μ_d, and variance, σ_d^2, of d, assuming x and y are independent.

Solution: d=y-x

$$E(d) = E(y - x) = E(y) - E(x) = 19.80 - 10.13 = 9.67$$

$$\sigma_{aX+bY}^2 = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\sigma_{XY}$$

$$\sigma_d^2 = \sigma_{y-x}^2 = \sigma_y^2 + (-1)^2\sigma_x^2 + 2(-1)\sigma_{XY} = 1.24 + 1.19 - 2(0) = 2.43$$

(3.5) Find the covariance of x and d, σ_{xd}.

Solution:

$$\sigma_{xd} = E[XD] - E[X]E[D] = 96.60 - 10.13(9.67) = -1.357$$

(3.6) Find the correlation coefficients, ρ_{xd}.

Solution:

$$\rho_{xd} = \frac{\sigma_{xd}}{\sigma_x\sigma_d} = \frac{-1.357}{\sqrt{1.19}\sqrt{2.43}} = -0.798$$

(3.7) Explain the sign of ρ_{xd}.

Solution:

The sign of ρ_{xd} is negative because from the definition of d, as x increases, d decreases.

(4.1) For the joint PDF, $f(x,y) = \begin{cases} \frac{1}{3}(x - y) & \text{for } 2 \le x \le 4 \text{ and } 1 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$, find the marginal distribution, $g(x)$.

Solution:

$$g(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_{1}^{2} \frac{1}{3}(x - y)dy = \frac{x}{3} - \frac{1}{2}$$

(4.2) Find $P(y \le 1.5) \text{ when } x = 4$, namely $f(x = 4, y \le 1.5)$

$$f(x = 4, y \le 1.5) = \int_{1}^{1.5} f(x = 4, y)dy = \int_{1}^{1.5} \frac{1}{3}(4 - y)dy = 0.4583$$

(4.3) Find the conditional probability $P(y \le 1.5 \mid x = 4)$ using the joint PDF from problem (12).

$$P(y \le 1.5 \mid x = 4) = f(y|x) = \frac{f(x = 4, y \le 1.5)}{g(x = 4)} = \frac{0.4583}{0.8333} = 0.55$$