Exam III Solutions
Administered: Wednesday, November 4, 2026
18 points

For each problem part: 0 points if not attempted or no work shown,
1 point for partial credit, if work is shown,
2 points for correct numerical value of solution

**Problem 1 (10 points)**

Consider the ternary Au-Pb-Sn alloy at the eutectic point. The compositions of the four phases present at the eutectic point are given in the table below. Additionally, the total composition of the alloy is given in the last row of the table.

| phase | Au (at%) | Pb (at%) | Sn (at%) |
|---|---|---|---|
| α (Pb rich) | 0.0 | 0.71 | 0.29 |
| β (Sn rich) | 0.0 | 0.02 | 0.98 |
| γ ($AuSn_4$) | 0.2 | 0.0 | 0.8 |
| L (liquid) | 0.03 | 0.251 | 0.719 |
| T (total) | 0.08 | 0.212 | 0.708 |

The phase fractions must sum to unity.

$$f_\alpha + f_\beta + f_\gamma + f_L = 1 \tag{1}$$

The mass balances on each component are written as follows

$$f_\alpha x_{Au}^\alpha + f_\beta x_{Au}^\beta + f_\gamma x_{Au}^\gamma + f_L x_{Au}^L = x_{Au}^T \tag{2}$$

$$f_\alpha x_{Pb}^\alpha + f_\beta x_{Pb}^\beta + f_\gamma x_{Pb}^\gamma + f_L x_{Pb}^L = x_{Pb}^T \tag{3}$$

$$f_\alpha x_{Sn}^\alpha + f_\beta x_{Sn}^\beta + f_\gamma x_{Sn}^\gamma + f_L x_{Sn}^L = x_{Sn}^T \tag{4}$$

In this set of equations, the phase fractions, $f_\alpha$, $f_\beta$ and $f_\gamma$ are unknown. The phase fraction of the liquid, $f_L$, is measured to be $f_L = 0.1$

(a) Are these equations linear or non-linear?
(b) Rearrange the system of equations into the form of $\underline{Ax} = \underline{b}$. Identify each variable.
(c) Provide the determinant of the matrix.
(d) How many solutions are there to this system of equations?
(e) Determine the remaining set of phase fractions, $f_\alpha$, $f_\beta$ and $f_\gamma$.

**Solution:**

(a) Are these equations linear or non-linear?

The equations are linear. The unknowns are multiplied by constants and added together.

(b) Rearrange the system of equations into the form of $\underline{Ax} = \underline{b}$. Identify each variable.

This problem has three unknown variables. Therefore we require three independent equations. In putting the equations into the form of $\underline{Ax} = \underline{b}$, all constant terms must be moved to the right-hand side in the $\underline{b}$ vector. Because The phase fraction of the liquid, $f_L$, is given as a constant, it must be moved to the right-hand side. The equations can be rearranged as

$$f_\alpha + f_\beta + f_\gamma = 1 - f_L \tag{1}$$

$$f_\alpha x_{Au}^\alpha + f_\beta x_{Au}^\beta + f_\gamma x_{Au}^\gamma = x_{Au}^T - f_L x_{Au}^L \tag{2}$$

$$f_\alpha x_{Pb}^\alpha + f_\beta x_{Pb}^\beta + f_\gamma x_{Pb}^\gamma = x_{Pb}^T - f_L x_{Pb}^L \tag{3}$$

$$f_\alpha x_{Sn}^\alpha + f_\beta x_{Sn}^\beta + f_\gamma x_{Sn}^\gamma = x_{Sn}^T - f_L x_{Sn}^L \tag{4}$$

You can use any three of the four equations. Here we choose the last three.

$$\underline{\mathrm{A}} = \begin{bmatrix} x_{Au}^\alpha & x_{Au}^\beta & x_{Au}^\gamma \\ x_{Pb}^\alpha & x_{Pb}^\beta & x_{Pb}^\gamma \\ x_{Sn}^\alpha & x_{Sn}^\beta & x_{Sn}^\gamma \end{bmatrix} \tag{4}$$

$$\underline{b} = \begin{bmatrix} x_{Au}^T - f_L x_{Au}^L \\ x_{Pb}^T - f_L x_{Pb}^L \\ x_{Sn}^T - f_L x_{Sn}^L \end{bmatrix} \tag{5}$$

$$\underline{x} = \begin{bmatrix} f_\alpha \\ f_\beta \\ f_\gamma \end{bmatrix} \tag{6}$$

(c) Provide the determinant of the matrix of elastic constants.

Per the Python code provided below, the determinant of the matrix of elastic constants is

Determinant of A: 0.138

(d) How many solutions are there to this system of equations?

Because the determinant is not zero, there is one unique solution to the system of equations.

(e) Determine the remaining set of phase fractions, $f_\alpha$, $f_\beta$ and $f_\gamma$.

Per the Python code provided below, the vector

$$\underline{x} = \begin{bmatrix} f_\alpha \\ f_\beta \\ f_\gamma \end{bmatrix} = \begin{bmatrix} 0.25594203 \\ 0.25905797 \\ 0.385 \end{bmatrix}$$

*** python script start ***

```
import numpy as np

# Define A matrix and b vector
n = 3
Amat = np.zeros((n,n))
bvec = np.zeros((n,1))

# second equation - mass balance on Au
xAu_alpha = 0.0
xAu_beta = 0.0
xAu_gamma = 0.2
xAu_liquid = 0.03
xAu_total = 0.08

Amat[0,0] = xAu_alpha
Amat[0,1] = xAu_beta
Amat[0,2] = xAu_gamma

f_liquid = 0.1
bvec[0] = xAu_total - f_liquid*xAu_liquid

# third equation - mass balance on Pb
xPb_alpha = 0.71
xPb_beta = 0.02
xPb_gamma = 0.0
xPb_liquid = 0.251
xPb_total = 0.212

Amat[1,0] = xPb_alpha
Amat[1,1] = xPb_beta
Amat[1,2] = xPb_gamma

bvec[1] = xPb_total - f_liquid*xPb_liquid

# fourth equation - mass balance on Sn
xSn_alpha = 0.29
xSn_beta = 0.98
xSn_gamma = 0.8
xSn_liquid = 0.719
xSn_total = 0.708

Amat[2,0] = xSn_alpha
Amat[2,1] = xSn_beta
Amat[2,2] = xSn_gamma

bvec[2] = xSn_total -f_liquid*xSn_liquid
```

```
print(f"A matrix: {Amat}")
print(f"b vector: {bvec}")

# Compute determinant of Amat
detA = np.linalg.det(Amat)
print(f"Determinant of A: {detA}")

# Compute inverse of Amat
invA = np.linalg.inv(Amat)
# print(f"(f) Inverse of A:\n{invA}")

# Solve Ax = b
xvec = invA@bvec
print(f"Solution to A*x = b: {xvec}")

f_alpha = xvec[0]
f_beta = xvec[1]
f_gamma = xvec[2]

print(f" f_alpha = {f_alpha}")
print(f" f_beta = {f_beta}")
print(f" f_gamma = {f_gamma}")
print(f" f_liquid = {f_liquid}")
```

*** python script end ***

When run, this script yields the following output:

```
run xm03_p01_v02.py
A matrix: [[0.   0.   0.2 ]
 [0.71 0.02 0.  ]
 [0.29 0.98 0.8 ]]
b vector: [[0.077 ]
 [0.1869]
 [0.6361]]
Determinant of A: 0.13799999999999998
Solution to A*x = b: [[0.25594203]
 [0.25905797]
 [0.385     ]]
 f_alpha = [0.25594203]
 f_beta = [0.25905797]
 f_gamma = [0.385]
 f_liquid = 0.1
```

**Problem 2. (8 points)**

The Hall-Petch equation states that the yield strength, $\sigma_y$, varies with grain size, $d$, according to

$$\sigma_y = \sigma_o + k_y \frac{1}{\sqrt{d}} \tag{7}$$

where $\sigma_o$ and $k_y$ are material-specific parameters.
[Materials Science and Engineering: An Introduction, 5th Edition, William D. Callister, John Wiley & Sons, Inc., New York, 2000, p. 167.]

For the $\sigma_y$ vs $d$ data given in the file, http://utkstair.org/clausius/docs/mse301/data/xm3p02_f26.txt, perform the following tasks. The grain size is given in mm and the yield strength in MPa.

(a) Identify all variables, $y = mx + b$, when equation (7) is linearized.
(b) Report the best value of $\sigma_o$ and $k_y$.
(c) Report the standard deviations of $\sigma_o$ and $k_y$.
(d) Report the measure of fit.

**Solution**

(a) Identify all variables, $y = mx + b$, when equation (1) is linearized.

$y = \sigma_y$, $x = \frac{1}{\sqrt{d}}$, $m = k_y$, $b = \sigma_o$

(b) Report the best value of $\sigma_o$ and $k_y$.
(c) Report the standard deviations of $\sigma_o$ and $k_y$.
(d) Report the measure of fit.

I wrote a Python script to perform these three tasks in (b), (c) and (d). The script calls the linreg1.py function to perform the regression.

*** python script start ***

```
import numpy as np
import matplotlib.pyplot as plt
from linreg1 import linreg1

# read data
datamat = np.loadtxt('xm3p02_f26.txt')
ndata= datamat.shape[0]
print(f"There are {ndata} data points.")
grainsize = datamat[:,0]
yieldstrength = datamat[:,1]

# plot data
plt.plot(grainsize, yieldstrength, 'k-')
plt.xlabel('grain size (mm)')
plt.ylabel('yield strength (MPa)')

# linearize the equations
```

```
xreg = np.zeros(ndata)
yreg = np.zeros(ndata)
for i in range(ndata):
    xreg[i] = 1.0 / np.sqrt(grainsize[i])
    yreg[i] = yieldstrength[i]


# perform regression
(b0, b1), (b0_sd, b1_sd), MOF = linreg1(xreg,yreg)

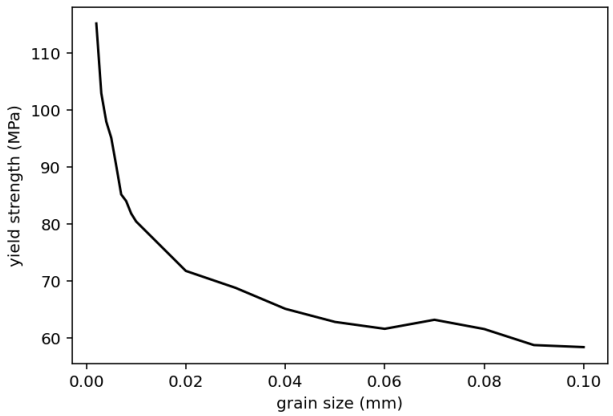
sigmao = b0
ky = b1
print(f"sigmao = {sigmao}")
print(f"ky = {ky}")
sigmaosd = b0_sd
kysd = b1_sd
print(f"sigmao standard deviation = {sigmaosd}")
print(f"ky standard deviation s= {kysd}")
print(f"MOF = {MOF}")
```

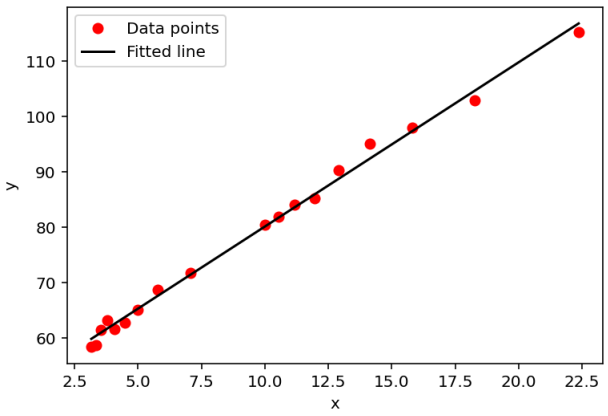
*** python script end ***

When run, this script yields the following output:

```
There are 18 data points.
sigmao = 50.48945287189166
ky = 2.9648706449315663
sigmao standard deviation = 0.5919746973570167
ky standard deviation s= 0.05452309668633954
MOF = 0.9946182202401529
```

It also generates two plots:




Thus

(b) Report the best value of $\sigma_o$ and $k_y$.

$\sigma_o = 50.49$ MPa $\quad k_y = 2.96$ MPa·m$^{½}$

(c) Report the standard deviations of $\sigma_o$ and $k_y$.

The standard deviations of these two parameters are

$\sigma_o = 0.59$ MPa $\quad k_y = 0.055$ MPa·m$^{½}$

(d) Report the measure of fit.

MOF = 0.995