Exam I Solutions
Administered: Wednesday, September 15, 2021
34 points

For each problem part:
0 points if not attempted or no work shown,
1 point for partial credit, if work is shown,
2 points for correct numerical value of solution

Problem 1. (12 points)

Consider the data for the following **24** metals given below. This data is available electronically on the course website.

	electric conductivity (10^6 Siemens/m)	thermal conductivity (W/m.K)	density (g/cm^3)	melting or deterioration temperature (°C)
Silver	62.1	420	10.5	961
Copper	58.7	386	8.9	1083
Gold	44.2	317	19.4	1064
Aluminum	36.9	237	2.7	660
Molybdenum	18.7	138	10.2	2623
Zinc	16.6	116	7.1	419
Lithium	10.8	84.7	0.54	181
Brass	15.9	150	8.5	900
Nickel	14.3	91	8.8	1455
Steel	10.1	80	7.9	1528
Palladium	9.5	72	12	1555
Platinum	9.3	107	21.4	1772
Tungsten	8.9	174	19.3	3422
Tin	8.7	67	7.3	232
Bronze 67Cu33Sn	7.4	85	8.8	1040
Carbone steel	5.9	54	7.7	1400
Carbone	5.9	129	1.8	2500
Lead	4.7	35	11.3	327
Titanium	2.4	21	4.5	1668
Stainless steel 316L	1.32	15	7.9	1535
Stainless steel 304	1.37	16.3	7.9	1450
Stainless steel 310	1.28	14.2	7.75	2650
Mercury	1.1	8	13.5	39
FeCrAl	0.74	16	7.2	1440

Answer the following questions for the metals in this tables.

(a) Determine the mean thermal conductivity.
(b) Determine the mean electrical conductivity.
(c) Determine the standard deviation of the thermal conductivity.
(d) Determine the standard deviation of the electrical conductivity.
(e) Determine the correlation coefficient between the thermal conductivity and electrical conductivity.
(f) What is the physical significance of your answer to part (e)?

Solution:
(a) Determine the mean thermal conductivity.

$$\mu_k = \frac{\sum_{i=1}^{n} k_i}{n} = \frac{2833.2}{24} = 118.05 \ \frac{W}{m{\cdot}K}$$

The mean thermal conductivity of these 24 metals is $118.05\ \frac{W}{m \cdot K}$.

(b) Determine the mean electrical conductivity.

$$\mu_\gamma = \frac{\sum_{i=1}^{n} \gamma_i}{n} = \frac{356.81}{24} = 14.87x10^{\ 6}\frac{S}{m}$$

The mean electrical conductivity of these 24 metals is $14.87x10^{\ 6}\frac{S}{m}$.

(c) Determine the standard deviation of the thermal conductivity.

$$\sigma_k^2 = E[k^2] - E[k\ \ ]^2 = \frac{639767}{24} - (118.05)^2 = 12721.17\left(\frac{W}{m \cdot K}\right)^2$$

$$\sigma_k^2 = \sqrt{\sigma_k^2} = 112.79\ \frac{W}{m \cdot K}$$

The standard deviation of the thermal conductivity is $112.79\ \frac{W}{m \cdot K}$.

(d) Determine the standard deviation of the electrical conductivity.

$$\sigma_\gamma^2 = E[\gamma^2] - E[\gamma\ \ ]^2 = \frac{12409}{24} - (14.87)^2 = 296.03\left(x10^{\ 6}\frac{S}{m}\right)^2$$

$$\sigma_\gamma^2 = \sqrt{\sigma_\gamma^2} = 17.21\ x10^{\ 6}\frac{S}{m}$$

The standard deviation of the electrical conductivity is $17.21\ x10^{\ 6}\frac{S}{m}$.

(e) Determine the correlation coefficient between the thermal conductivity and electrical conductivity.

$$\sigma_{k \cdot \gamma} = E[k \cdot \gamma] - E[k]E[\gamma] = \frac{87227}{24} - 118.05\ \cdot 14.87 = 2453.77x10^{\ 6}\frac{W \cdot S}{m^2 \cdot K}$$

$$\rho_{k \cdot T_m} = \frac{\sigma_{k \cdot \gamma}}{\sigma_k \cdot \sigma_\gamma} = \frac{1879.41}{112.79 \cdot 17.21} = 0.9685$$

The correlation coefficient of thermal conductivity and electrical conductivity is 0.9685.

(f) What is the physical significance of your answer to part (e)?

A positive correlation coefficient indicates that, from a statistical point of view, as the thermal conductivity increases, the electrical conductivity increases (or equivalently as the thermal conductivity decreases, the electrical conductivity decreases).

Problem 2. (10 points)

Today, public health decisions in the United States are impacted by political affiliation. We can evaluate this issue by examining the party affiliation of a state's governor and the presence of colleges or universities within the state that require a COVID-19 vaccine. Currently of the 50 states, 27 have Republican governors and 23 have Democratic governors. The following information is valid as of September 10, 2021. We report the number of states with at least one college or university with a COVID-19 vaccine mandate.

States with Republican Governor & no college w/ Vaccine Mandate	9	(R & 0)
States with Republican Governor & at least one college w/ Vaccine Mandate	18	(R & 1)
States with Democratic Governor & no college w/ Vaccine Mandate	1	(D & 0)
States with Democratic Governor & at least one college w/ Vaccine Mandate	22	(D & 1)

Using this information, answer the following questions.

(a) Draw a Venn Diagram of the sample space for this data.
(b) What is the probability that a state has at least one college or university with a vaccine mandate?
(c) What is the probability that a state had no university with a vaccine mandate given that it had a Republican governor?
(d) What is the probability that a state had no university with a vaccine mandate given that it had a Democratic governor?
(e) What is the probability that a state had a Republican governor given that no university had a vaccine mandate?
(f) Prove that the party affiliation of the governor and the presence of mandates are not independent of each other.

Solution:

There are a total of 9+18+1+22 = 50 states.
We are given:

$$P(R \cap 0) = \frac{9}{50}$$
$$P(R \cap 1) = \frac{18}{50}$$
$$P(D \cap 0) = \frac{1}{50}$$
$$P(D \cap 1) = \frac{22}{50}$$

(a) Draw a Venn Diagram of the sample space for this experiment.

$R \cap 0$	$R \cap 1$
$D \cap 0$	$R \cap 1$

(b) What is the probability that a state has at least one college or university with a vaccine mandate?

We can use the union rule.

$$P(1) = P(R \cap 1) + P(D \cap 1) - P[(R \cap 1) \cap (D \cap 1)] = \frac{18}{50} + \frac{22}{50} - 0 = \frac{40}{50} = \frac{4}{5}$$

The probability that a state has at least one college or university with a vaccine mandate is 0.8.

(c) What is the probability that a state had no university with a vaccine mandate given that it had a Republican governor?

Consider the conditional probability rule.

$$P(0|R) = \frac{P(R \cap 0)}{P(R)} = \frac{9}{50} \Big/ \frac{27}{50} = \frac{1}{3} = 0.333$$

The probability that a state had no university with a vaccine mandate given that it had a Republican governor is 0.333.

(d) What is the probability that a state had no university with a vaccine mandate given that it had a Democratic governor?

Consider the conditional probability rule.

$$P(0|D) = \frac{P(D \cap 0)}{P(D)} = \frac{1}{50} \Big/ \frac{23}{50} = \frac{1}{23} = 0.0435$$

The probability that a state had no university with a vaccine mandate given that it had a Democratic governor is 0.0435.

(e) What is the probability that a state had a Republican governor given that no university had a vaccine mandate?

$$P(0) = P(R \cap 0) + P(D \cap 0) - P[(R \cap 0) \cap (D \cap 0)] = \frac{9}{50} + \frac{1}{50} - 0 = \frac{10}{50} = \frac{1}{5}$$

$$P(R|0) = \frac{P(R \cap 0)}{P(0)} = \frac{9}{50} \Big/ \frac{10}{50} = \frac{9}{10} = 0.9$$

The probability that a state had a Republican governor given that no university had a vaccine mandate is 0.9.

(f) Prove that the party affiliation of the governor and the presence of mandates are not independent of each other.

There are a variety of ways to prove this.

If A & B are independent, then $P(A|B) = P(A)$ or $P(B|A) = P(B)$.

So for example, we can show that party affiliation of the governor and the presence of mandates are not independent of each other through any of the following statements.

$$P(0|R) = \frac{1}{3} \neq P(0) = \frac{1}{5}$$

$$P(0|D) = \frac{1}{23} \neq P(0) = \frac{1}{5}$$

$$P(R|0) = \frac{9}{10} \neq P(R) = \frac{27}{50}$$

Problem 3. (10 points)

Consider the following probability density function.

$$f(x) = \begin{cases} c(x^3 + x) & for\ 0 < x < 3 \\ 0 & otherwise \end{cases}$$

(a) Is the PDF continuous or discrete?

(b) Find the value of *c* that normalizes this PDF.
(c) Find the probability that x is less than 1.0.
(d) Find the probability that x is greater than 1.0.
(e) Find the average value of the random variable x.

Solution:

(a) Is the PDF continuous or discrete?

This PDF is continuous.

(b) Find the value of c that normalizes this PDF.

$$1=\int_0^3 f(x)\,dx = c\int_0^2 (x^3+x)\,dx = c\left[\frac{x^4}{4}+\frac{x^2}{2}\right]_0^3 = c\left(\frac{81}{4}+\frac{9}{2}\right) = c\,\frac{99}{4}$$

The value of *c* that normalizes this PDF is $c = \frac{4}{99} \sim 0.0404$.

(c) Find the probability that x is less than 1.0.

$$P(x<1.0) = \int_0^{1.0} f(x)\,dx = \frac{4}{99}\int_0^1 (x^3+x)\,dx = \frac{4}{99}\left[\frac{x^4}{4}+\frac{x^2}{2}\right]_0^1 = \frac{4}{99}\left(\frac{1}{4}+\frac{1}{2}\right) = \frac{3}{99} \sim 0.0303$$

(d) Find the probability that x is greater than 1.0.

$$P(x>1.0) = 1 - P(x<1.0) = 1 - \frac{3}{99} = \frac{96}{99} \sim 0.970$$

(e) Find the average value of the random variable x.

$$E[x] = \int_0^3 x f(x)\,dx = \frac{4}{99}\int_0^3 x(x^3+x)\,dx$$
$$= \frac{4}{99}\left[\frac{x^5}{5}+\frac{x^3}{3}\right]_0^3 = \frac{4}{99}\left[\frac{243}{5}+\frac{27}{3}\right] = \frac{4}{99}\left[\frac{864}{15}\right] = \frac{4}{99}\left[\frac{288}{5}\right] = \frac{1152}{495} \sim 2.327$$