Exam I
Administered: Wednesday, September 19, 2018
34 points

For each problem part: 0 points if not attempted or no work shown,
1 point for partial credit, if work is shown,
2 points for correct numerical value of solution

Problem 1. (14 points)

Consider the data for the following 24 metals given below, taken from https://www.tibtech.com/conductivite.php

	Electric conductivity (10^6 S/m)	Electric resistivity (10^{-8} Ω.m)	Thermal conductivity (W/m.K)	Thermal expansion $10^{-6}(K^{-1})$	Density (g/cm^3)	Melt/decompose temperature (°C)
Silver	62.1	1.6	420	19.1	10.5	961
Copper	58.7	1.7	386	17	8.9	1083
Gold	44.2	2.3	317	14.1	19.4	1064
Aluminium	36.9	2.7	237	23.5	2.7	660
Molybdenum	18.7	5.34	138	4.8	10.2	2623
Zinc	16.6	6	116	31	7.1	419
Lithium	10.8	9.3	84.7	56	0.54	181
Brass	15.9	6.3	150	20	8.5	900
Nickel	14.3	7	91	13.3	8.8	1455
Steel	10.1	9.9	80	12.1	7.9	1528
Palladium	9.5	10.5	72	11	12	1555
Platinium	9.3	10.8	107	9	21.4	1772
Tungsten	8.9	11.2	174	4.5	19.3	3422
Tin	8.7	11.5	67	23.5	7.3	232
Bronze 67Cu33Sn	7.4	13.5	85	17	8.8	1040
Carbone steel	5.9	16.9	54	12	7.7	1400
Carbone	5.9	16.9	129	0.2	1.8	2500
Lead	4.7	21.3	35	29	11.3	327
Titanium	2.4	41.7	21	8.9	4.5	1668
Stainless steel 316L EN1.4404	1.32	76	15	16.5	7.9	1535
Stainless steel 304 EN1.4301	1.37	73	16.3	16.5	7.9	1450
Stainless steel 310 EN1.4841	1.28	78	14.2	17	7.75	2650
Mercury	1.1	90.9	8	61	13.5	-39
FeCrAl	0.74	134	16	11.1	7.2	-1440

Based on this data, the following data analysis has been performed.

property	symbol	mean
Electric conductivity (10^6 S/m)	sigma	14.86708333
Electric resistivity (10^{-8} W.m)	r	27.43083333
Thermal conductivity (W/m.K)	k	118.05
Thermal expansion $10^{-6}(K^{-1})$	beta	18.67083333
Density (g/cm^3)	rho	9.287083333
Melt/decompose temperature (°C)	Tmelt	1326.083333

mean of the square (diagonal) or product (off-diagonal)						
	sigma	r	k	beta	rho	Tmelt
sigma	517.0598	99.97283	3634.471	268.4808	148.8138	17058.8
r		1966.424	1003.868	583.3884	238.24	21767.89
k			26656.98	1985.823	1211.977	151556.9
beta				543.9029	160.8229	14499.41
rho					111.4406	12403.41
Tmelt						2461391

variance (diagonal) and covariance (off-diagonal)						
	sigma	r	k	beta	rho	Tmelt
sigma	296.0296373	-307.8436517	1879.411938	-9.100043403	10.74194983	-872.1414236
r		1213.973366	-2234.341958	71.23189931	-16.51243507	-11315.98257
k			12721.17333	-218.2689583	115.6372292	9178.808333
beta				195.3028993	-12.5746684	-8019.168403
rho					25.19067066	1202.413576
Tmelt						1006753.91

correlation coefficient						
	sigma	r	k	beta	rho	Tmelt
sigma	1	-0.5135208	0.96848	-0.03785	0.124393	-0.050519
r		1	-0.56857	0.14629	-0.09442	-0.323688
k			1	-0.13848	0.204275	0.0811075
beta				1	-0.17928	-0.571891
rho					1	0.2387659
Tmelt						1

Answer the following questions.

(a) Determine the variance of the electric conductivity. Include units.
(b) Determine the standard deviation of the thermal conductivity. Include units.
(c) Determine the covariance of electric conductivity and electric resistivity. Include units.
(d) Determine the correlation coefficient of thermal expansion coefficient and melt temperature.
(e) Are the electrical conductivity and thermal conductivity independent variables? Justify answer.
(f) What property is most strongly correlated with density?
(g) Based on your understanding of physics, should the electrical conductivity be positively or negatively correlated with electrical resistivity? Justify answer.

(a) Determine the variance of the electric conductivity. Include units.

$$\sigma_{sigma}^2 = E[sigma^2] - E[sigma]^2 = 517.0598 - 14.86708^2 = 296.0296373$$

The variance of the electric conductivity is 296.0 $(10^6$ S/m$)^2$.

(b) Determine the standard deviation of the thermal conductivity. Include units.

$$\sigma_k = \sqrt{\sigma_k^2} = \sqrt{12721.17333} = 112.78817$$

The standard deviation of the thermal conductivity is 112.8 (W/m.K).

(c) Determine the covariance of electric conductivity and electric resistivity. Include units.

$$\sigma_{sigma \cdot r} = E[sigma \cdot r] - E[sigma]E[r] = 99.97283 - 14.86708 \cdot 27.43083 = -307.8436517$$

The covariance of electric conductivity and electric resistivity is –307.8 10^6 S/m·10^{-8} Ω.m. These units can be reduced to –307.8 10^{-2}. The Siemen is the inverse of the Ohm, so they cancel. The covariance is thus dimensionless.

(d) Determine the correlation coefficient of thermal expansion coefficient and melt temperature.

$$\rho_{beta \cdot Tmelt} = \frac{\sigma_{beta \cdot Tmelt}}{\sqrt{\sigma_{beta}^2} \cdot \sqrt{\sigma_{Tmelt}^2}} = \frac{-8019.168403}{\sqrt{195.3028993} \cdot \sqrt{1006753.91}} = -0.571891$$

The correlation coefficient of thermal expansion coefficient and melt temperature is -0.572.

(e) Are the electrical conductivity and thermal conductivity independent variables? Justify answer.

The correlation coefficient of the electric conductivity and the thermal conductivity is 0.96848. Therefore, these two variables are not independent. To the contrary, they are strongly positively correlated.

(f) What property is most strongly correlated with density?

The correlation coefficient with the largest magnitude involving density is $\rho_{rho \cdot Tmelt}$ with a value of 0.2387659. Thus the melting temperature is most strongly correlated with density.

(g) Based on your understanding of physics, should the electrical conductivity be positively or negatively correlated with electrical resistivity? Justify answer.

The electrical conductivity is the inverse of the electrical resistivity. Both properties are positive numbers. So as one increases, the other must decrease. In this case, they must be negatively correlated. For this set of materials, the correlation coefficient between these two properties is -0.514.

Problem 2. (10 points)

A study is performed involving lithium ion batteries manufactured with a flake graphite (G) or lignin-based (L) anode. Each battery is evaluated for one hundred cycles and is categorized as either Pass (P) or Fail (F).

During this study the following information was collected.

# of batteries with a flake graphite anode that passed the test = 19
# of batteries with a flake graphite anode that failed the test = 6
# of batteries with a lignin-based anode that passed the test = 13
# of batteries with a lignin-based anode that failed the test = 2

Using this information, answer the following questions.

(a) Draw a Venn Diagram of the sample space for this experiment.
(b) What is the probability that a battery in this study contained a flake graphite anode?
(c) What is the probability that a battery passed the test given that it contained a flake graphite anode?
(d) What is the probability that a battery passed the test given that it contained a lignin-based anode?
(e) What is the probability that the battery contained the lignin-based anode given that the test failed?

Solution:

G = flake Graphite anode
L = Lignin-based anode
P = pass
F = fail

There are a total of 19+6+13+2 = 40 batteries
We are given:

$$P(G \cap P) = \frac{19}{40}$$
$$P(G \cap F) = \frac{6}{40}$$
$$P(L \cap P) = \frac{13}{40}$$
$$P(L \cap F) = \frac{2}{40}$$

(a) Draw a Venn Diagram of the sample space for this experiment.

$G \cap P$	$G \cap F$
$L \cap P$	$L \cap F$

(b) What is the probability that a battery in this study contained a flake graphite anode?

We can use the union rule.

$$P(G) = P(G \cap P) + P(G \cap F) - P[(G \cap P) \cap (G \cap F)] = \frac{19}{40} + \frac{6}{40} - 0 = \frac{25}{40}$$

The probability that a battery had a flake graphite anode in this study is 0.625.

(c) What is the probability that a battery passed the test given that it contained a flake graphite anode?

Consider the conditional probability rule.

$$P(P|G) = \frac{P(G \cap P)}{P(G)} = \frac{\frac{19}{40}}{\frac{25}{40}} = \frac{19}{25}$$

The probability that a battery passed the test given that it contained a flake graphite anode is 0.76.

(d) What is the probability that a battery passed the test given that it contained a lignin-based anode?

First calculate the probability that a battery has a lignin-based anode.

$$P(L) = 1 - P(G) = 1 - \frac{25}{40} = \frac{15}{40}$$

Use the conditional probability rule

$$P(P|L) = \frac{P(L \cap P)}{P(L)} = \frac{\frac{13}{40}}{\frac{15}{40}} = \frac{13}{15}$$

The probability that a battery passed the test given that it contained a lignin-based anode is 0.867.

(e) What is the probability that the battery contained the lignin-based anode given that the test failed?

$$P(F) = P(F \cap G) + P(F \cap L) - P[(F \cap G) \cap (F \cap L)] = \frac{6}{40} + \frac{2}{40} - 0 = \frac{8}{40}$$

Use the conditional probability rule

$$P(L|F) = \frac{P(L \cap F)}{P(F)} = \frac{\frac{2}{40}}{\frac{8}{40}} = \frac{1}{4}$$

The probability that a battery contained a lignin-based anode given that it failed the test is 0.25.

Problem 3. (10 points)

Consider the following probability density functiton.

$$f(x) = \begin{cases} cx^4 & for\ 0 < x < 1 \\ 0 & otherwise \end{cases}$$

(a) Is the PDF continuous or discrete?
(b) Find the value of k that normalizes this PDF.
(c) Find the probability is less than 0.2.
(d) Find the probability is greater than 0.2.
(e) Find the average value of the random variable x.

Solution:

(a) Is the PDF continuous or discrete?

This PDF is continuous.

(b) Find the value of c that normalizes this PDF.

$$1=\int_0^1 f(x)\,dx = c\int_0^1 x^4\,dx = c\left[\frac{x^5}{5}\right]_0^1 = c\frac{1}{5}$$

The value of c that normalizes this PDF is c = 5.

(c) Find the probability is less than 0.2.

$$P(x < 0.2) = \int_0^{0.2} f(x)\,dx = 5\int_0^{0.2} x^4\,dx = 5\left[\frac{x^5}{5}\right]_0^{0.2} = 5\frac{0.2^5}{5} = 0.00032$$

(d) Find the probability is greater than 0.2.

$$P(x > 0.2) = 1 - P(x < 0.2) = 1 - 0.00032 = 0.99968$$

(e) Find the average value of the random variable x.

$$E[x] = \int_0^1 xf(x)\,dx = 5\int_0^1 x\cdot x^4\,dx = 5\left[\frac{x^6}{6}\right]_0^1 = 5\frac{1}{6} = \frac{5}{6}$$