Exam II Solutions
Administered: Friday, October 13, 2016
28 points

For each problem part: 0 points if not attempted or no work shown,
1 point for partial credit, if work is shown,
2 points for correct numerical value of solution

Problem 1. (20 points)

The National Institute of Standards and Technology (NIST) maintains a variety of websites providing physical properties. The NIST Chemistry Webbook, http://webbook.nist.gov/chemistry/, reports various properties for numerous compounds. Where multiple entries exist for a single property of a given compound, NIST reports an average. For example, consider the critical temperature of benzene. NIST notes that from 1881 to 1995, this property has been reported by 41 researchers. Rather than reproduce the table in its entirety, here we present a few of the newest and oldest values and the following information.

Tc (K)	Reference
562.05 ± 0.07	Tsonopoulos and Ambrose, 1995
561.8	Chirico and Steele, 1994
562.2	Knipmeyer, Archer, et al., 1989
...	...
569.6	Schmidt, 1891
561.65	Young, 1889
564.9	Ramsay, 1881

$$\sum_{i=1}^{41} T_{C,i} = 23060.10 \text{ K} \quad \text{and} \quad \sum_{i=1}^{41} T_{C,i}^2 = 12970033.9686 \text{ K}^2$$

Perform the following tasks.

(a) Determine the sample mean of the critical temperature of benzene.
(b) Determine the sample variance of the critical temperature of benzene.
(c) Determine the sample standard deviation of the critical temperature of benzene.
(d) Identify the appropriate distribution to describe the mean of the critical temperature of benzene in this case.
(e) Determine the lower limit of a 98% confidence interval on the mean of the critical temperature of benzene.
(f) Determine the upper limit of a 98% confidence interval on the mean of the critical temperature of benzene.
(g) Identify the appropriate distribution to describe the variance of the critical temperature of benzene in this case.
(h) Determine the lower limit of a 98% confidence interval on the variance of the critical temperature of benzene.
(i) Determine the upper limit of a 98% confidence interval on the variance of the critical temperature of benzene.
(j) Explain your findings in parts (a) through (i) in language a non-statistician can understand.

Solution:

(a) Determine the sample mean of the critical temperature of benzene.

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{7} x_i = \frac{23060.10}{41} = 562.4414634 \text{ K}$$

Based on this data set the mean of the critical temperature of benzene is 562.44 K.

$$s^2 = \frac{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}{n(n-1)} = \frac{n\sum_{i=1}^{n} T_{c,i}^2 - \left(\sum_{i=1}^{n} T_{c,i}\right)^2}{n(n-1)} = \frac{41(12970033.9686) - (23060.10)^2}{41(41-1)}$$

$$= 1.939452805 \text{ K}^2$$

Based on this data set the sample variance of the critical temperature of benzene is 1.94 K^2.

(c) Determine the sample standard deviation of the critical temperature of benzene.

$$s = \sqrt{s^2} = \sqrt{1.939452805} = 1.392642382 \text{ K}$$

Based on this data set the sample variance of the critical temperature of benzene is 1.39 K.

(d) Identify the appropriate distribution to describe the mean of the critical temperature of benzene in this case.

In this case we do not know the true population variance so the appropriate distribution of the sample mean is the **t** distribution.

(e) Determine the lower limit of a 98% confidence interval on the mean of the critical temperature of benzene.
(f) Determine the upper limit of a 98% confidence interval on the mean of the critical temperature of benzene.

$$C.I. = 1 - 2\alpha = 0.98$$

$$\alpha = \frac{1 - C.I.}{2} = \frac{1 - 0.98}{2} = 0.01$$

$$v = n - 1 = 40$$

The limits on the t-distribution are given by

```
>> t = icdf('t',0.01,40)

t =  -2.423256779334856
```

and for the upper limit

```
>> t = icdf('t',0.99,40)

t =   2.423256779334856
```

We next insert all of these numbers into the equation for the confidence interval on the sample mean.

$$P\left[\overline{X} + t_{\alpha}\frac{s}{\sqrt{n}} < \mu < \overline{X} + t_{1-\alpha}\frac{s}{\sqrt{n}}\right] = 1 - 2\alpha$$

$$P\left[562.44 - 2.4233\frac{1.39}{\sqrt{41}} < \mu < 562.44 + 2.4233\frac{1.39}{\sqrt{41}}\right] = 0.98$$

$$P[561.91 < \mu < 562.97] = 0.98$$

Based on this data set we are 98% confident that the true population mean lies between 561.91 K and 562.97 K.

(g) Identify the appropriate distribution to describe the variance of the critical temperature of benzene in this case.

The appropriate distribution of the sample variance is the χ^2 (chi-squared) distribution.

(h) Determine the lower limit of a 98% confidence interval on the variance of the critical temperature of benzene.
(i) Determine the upper limit of a 98% confidence interval on the variance of the critical temperature of benzene.

$$C.I. = 1 - 2\alpha = 0.98$$

$$\alpha = \frac{1 - C.I.}{2} = \frac{1 - 0.98}{2} = 0.01$$

$$v = n - 1 = 40$$

The limits on the χ^2-distribution are given by

```
>>  chi2 = icdf('chi2',0.01,40)

chi2 =  22.164261252975162
```

and for the upper limit

```
>> chi2 = icdf('chi2',0.99,40)

chi2 =  63.690739751564458
```

We next insert all of these numbers into the equation for the confidence interval on the sample variance.

$$P\left[\frac{(n-1)s^2}{\chi^2_{1-\alpha}} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{\alpha}}\right] = 1 - 2\alpha$$

$$P\left[\frac{(41-1)1.94}{63.691} < \sigma^2 < \frac{(41-1)1.94}{22.164}\right] = 0.98$$

$$P\left[1.2180 < \sigma^2 < 3.5001\right] = 0.98$$

Based on this data set we are 98% confident that the true population variances lies between 1.22 K^2 and 3.50 K^2.

(j) Explain your findings in parts (a) through (i) in language a non-statistician can understand.

Based on all 41 data points in the literature, we can make the following observations. The mean of the critical temperature of benzene is 562.44 K with a 98% confident interval in the range from 561.91 K to 562.97 K. The variance of the critical temperature of benzene is 1.94 K^2 with a 98% confident interval in the range from 1.22 K^2 to 3.50 K^2. (The NIST webbook reports a mean value of 562.0 ± 0.8 K because they discarded five of the 41 data points as outliers.)

Problem 2. (8 points)

A manufacturer of LED lightbulbs claims that the mean life of their batteries is μ = 50,000 hours with a standard deviation of σ = 5,000 hours. (Assume that the lifetime of a single bulb obeys the normal distribution.) There are 8760 hours in a year.

(a) What is the probability that a single bulb is still functioning after three years?
(b) What is the probability that a single bulb is still functioning after seven years?

You outfit your doomsday underground bunker with six lightbulbs.

(c) What is the probability that all six bulbs are functioning after three years?
(d) What is the probability that at least one bulb is functioning after seven years?

Solution:

(a) What is the probability that a single bulb is still functioning after three years?

$$P(x > 3) = 1 - P(x < 3) = 1 - \int_{-\infty}^{3} f(x;\mu,\sigma)dx$$

```
>> p = 1- cdf('normal',3*8760,50000,5000)
p = 0.999998952305761
```

There is a 99.9999% chance the bulb still functions after three years.

(b) What is the probability that a single bulb is still functioning after seven years?

$$P(x > 7) = 1 - P(x < 7) = 1 - \int_{-\infty}^{7} f(x;\mu,\sigma)dx$$

```
>> p = 1- cdf('normal',7*8760,50000,5000)
p =   0.011787057388953
```

There is a 1.18% chance the bulb still functions after seven years.

You outfit your doomsday underground bunker with six lightbulbs.

(c) What is the probability that all six bulbs are functioning after three years?

Use the binomial distribution to describe the distribution of the number of lights still functioning after a given period of time.

$$P(x = 6) = b(x = 6; n = 6, p = 0.999998952305761)$$

```
>> p = pdf('binomial',6,6,0.999998952305761)
p =   0.999993713851031
```

There is a 99.9994% chance that your trusty underground bunker remains perfectly lit after three years.

(d) What is the probability that at least one bulb is functioning after seven years?

Use the binomial distribution to describe the distribution of the number of lights still functioning after a given period of time.

$$P(x \geq 1) = 1 - P(x < 1) = 1 - P(x = 0) = 1 - b(x = 0; n = 6, p = 0.011787057388953)$$

```
>> p = 1 - pdf('binomial',0,6,0.011787057388953)

p =   0.068670787955773
```

OR

since for the binomial distribution

$$P(x = 0) = P(x \leq 0)$$

$$P(x \geq 1) = 1 - P(x < 1) = 1 - P(x \leq 0) = 1 - B(x = 0; n = 6, p = 0.011787057388953)$$

```
>> p = 1 - cdf('binomial',0,6,0.011787057388953)

p =   0.068670787955773
```

In either case, there is a 6.87% chance that your trusty underground bunker is not enclosed in total darkness after seven years.