Exam I
Administered: Wednesday, September 20, 2017
32 points

For each problem part: 0 points if not attempted or no work shown,
1 point for partial credit, if work is shown,
2 points for correct numerical value of solution

Problem 1. (14 points)

The Young's modulus and ultimate strength of thirteen metals are reported below.

Element	Young's modulus (GPa)	Ultimate strength (MPa)	modulus^2 (GPa^2)	Ultimate strength^2 (MPa^2)	modulus·strength (GPa·MPa)
silicon	107	7000	11449	49000000	749000
tungsten	411	585	168921	342225	240435
iron	211	350	44521	122500	73850
titanium	120	308	14400	94864	36960
copper	130	210	16900	44100	27300
tantalum	186	107	34596	11449	19902
tin	47	107	2209	11449	5029
zinc alloy	95	300	9025	90000	28500
nickel	170	168	28900	28224	28560
silver	83	170	6889	28900	14110
gold	79	100	6241	10000	7900
aluminium	70	45	4900	2025	3150
lead	16	12	256	144	192
sum	1725	9462	349207	49785880	1234888

Based on this data, answer the following questions.

(a) Find the average modulus of these thirteen metals.
(b) Find the variance of the ultimate tensile strength.
(c) Find the standard deviation of the ultimate tensile strength.
(d) Find the covariance of ultimate strength and modulus.
(e) Find the correlation coefficient of the ultimate strength and modulus.
(f) Are the ultimate strength and modulus independent random variables?
(g) What is the physical relationship between ultimate strength and modulus based on your answer to (g)?

Solution:

(a) Find the average modulus of these thirteen metals.

$$E[m] = \frac{1725}{13} = 133 \text{ GPa}$$

(b) Find the variance of the ultimate tensile strength.

$$\sigma_{uts}^2 = E\left[uts^2\right] - E[uts]^2 = \frac{49785880}{13} - \left(\frac{9462}{13}\right)^2 = 3299923 \text{ MPa}^2$$

(c) Find the standard deviation of the ultimate tensile strength.

$$\sigma_{uts} = \sqrt{\sigma_{uts}^2} = \sqrt{3299923} = 1817 \text{ MPa}$$

(d) Find the covariance of ultimate strength and modulus.

$$\sigma_{m \cdot uts} = E[m \cdot uts] - E[m]E[uts] = \frac{1234888}{13} - \frac{1725}{13} * \frac{9462}{13} = \text{-1588 GPa}\cdot\text{MPa}$$

(e) Find the correlation coefficient of the ultimate strength and modulus.

$$\sigma_m^2 = E\left[m^2\right] - E[m]^2 = \frac{349207}{13} - \left(\frac{1725}{13}\right)^2 = 9254.828402 \text{ GPa}^2$$

$$\sigma_m = \sqrt{\sigma_m^2} = \sqrt{9254.828402} = 96.2 \text{ GPa}$$

$$\rho_{m \cdot uts} = \frac{\sigma_{m \cdot uts}}{\sigma_m \sigma_{uts}} = \frac{-1588}{96.2 \cdot 1817} = \text{-0.0091}$$

(f) Are the ultimate strength and modulus independent random variables?

No, the ultimate strength and modulus are not independent random variables because the covariance is not zero.

(g) What is the physical relationship between ultimate strength and modulus based on your answer to (g)?

The correlation coefficient is very close to zero. Therefore, there is no strong relationship between the ultimate strength and the modulus.

Problem 2. (10 points)

In examining a set of specimens from two classes of materials—metals (m) and ceramics (c)—we note the following observations with respect to ductile (d) and brittle (b) properties.

In our sample set, the probability that a material ductile is 0.75.
The probability that a material is brittle given that it is ceramic is 0.95.
The probability that a material is a metal given that it is ductile is 0.993333.

Using this information, answer the following questions.

(a) Draw a Venn Diagram of the sample space for this experiment.
(b) What is the probability that a material is brittle?
(c) What is the probability that a material is metal and ductile?
(d) What is the probability that a material is ceramic and ductile?
(e) What is the probability that a material is ceramic?

Solution:

$$P(d) = 0.75$$
$$P(b|c) = 0.95$$
$$P(m|d) = 0.993333$$

(a) Draw a Venn Diagram of the sample space for this experiment.

$m \cap d$	$c \cap d$
$m \cap b$	$c \cap b$

(b) What is the probability that a material is brittle?

A material is either ductile or brittle, so those probabilities sum to one.

$$1 = P(d) + P(b)$$

$$P(b) = 1 - P(d) = 1 - 0.75 = 0.25$$

(c) What is the probability that a material is metal and ductile?

$$P(m \mid d) = \frac{P(m \cap d)}{P(d)}$$

$$P(m \cap d) = P(d)P(m \mid d) = 0.75 \cdot 0.99333 = 0.745$$

(d) What is the probability that a material is ceramic and ductile?

$$P(d) = P(m \cap d) + P(c \cap d)$$

$$P(c \cap d) = P(d) - P(m \cap d) = 0.75 - 0.745 = 0.005$$

(e) What is the probability that a material is ceramic?

$$P(c) = P(c \cap b) + P(c \cap d)$$

$$P(b \mid c) = \frac{P(c \cap b)}{P(c)}$$

This has two equation with two unknowns, $P(c)$ and $P(c \cap b)$. Rearranged the second equation for $P(c \cap b)$.

$$P(c \cap b) = P(c)P(b \mid c)$$

Substitute this intersection into the first equation.

$$P(c) = P(c \cap b) + P(c \cap d) = P(c)P(b \mid c) + P(c \cap d)$$

Solve for $P(c)$.

$$P(c) - P(c)P(b \mid c) = P(c \cap d)$$

$$P(c) = \frac{P(c \cap d)}{1 - P(b \mid c)} = \frac{0.005}{1 - 0.95} = 0.1$$

Problem 3. (8 points)

Consider the following PDF of the random variable, *x*,

$$f(x) = \begin{cases} cx^4 & for\ \ 0 \le \mathrm{x} \le 1 \\ 0 & otherwise \end{cases}$$

(a) Is the PDF continuous or discrete?
(b) Find the value of c that normalizes this PDF.
(c) Find the probability that x is greater than 0.25.
(d) Find the probability that x is less than 0.25.

Solution:

(a) Is this PDF continuous or discrete?

This PDF is continuous.

(b) Find the value of c that normalizes this PDF.

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} cx^4 dx = c\int_0^1 x^4 dx = c\left[\frac{x^5}{5}\right]_0^1 = c\left[\frac{1}{5} - \frac{0}{5}\right] = \frac{c}{5} = 1$$

$$c = 5$$

(c) Find the probability that x is greater that 0.25.

$$P(x > 0.25) = \int_{0.25}^{1} f(x)dx = c\int_{0.25}^{1} x^4 dx = c\left[\frac{x^5}{5}\right]_{0.25}^1 = c\left[\frac{1}{5} - \frac{0.25^5}{5}\right] = c\left[\frac{1}{5} - \frac{1}{5 \cdot 1024}\right]$$

$$= c\left[\frac{1023}{5 \cdot 1024}\right] = \frac{1023}{1024} \approx 0.999023$$

(d) Find the probability that x is less than 0.25.

$$P(x < 0.25) = 1 - P(x > 0.25) = 1 - \frac{1023}{1024} = \frac{1}{1024} \approx 0.000977$$