Exam II Solutions
Administered: Wednesday, October 5, 2016
24 points

For each problem part: 0 points if not attempted or no work shown,
1 point for partial credit, if work is shown,
2 points for correct numerical value of solution

Problem 1. (20 points)

Platinum is soluble to hydrogen. Consider the paper, "Solubility and Diffusion of Hydrogen and Deuterium in Platinum" by Y. Ebisuzaki, W. J. Kass and M. O'Keeffe, *J. Chem. Phys.* **49**, 3329 (1968); http://dx.doi.org/10.1063/1.1670604. This article quantifies the error in the diffusivity and activation energy. Let us perform our own investigation of the diffusivities of hydrogen and deuterium in platinum at 750 K based on the following set of physically reasonable but fictitious experimental data.

	Hydrogen			Deuterium	
	D	D^2		D	D^2
sample	(cm^2/s)	$(cm^2/s)^2$		(cm^2/s)	$(cm^2/s)^2$
1	0.000136492	1.86299E-08		0.000115868	1.34255E-08
2	9.28908E-05	8.62869E-09		9.37811E-05	8.79489E-09
3	0.000156639	2.45356E-08		9.90971E-05	9.82024E-09
4	0.000161559	2.61013E-08		9.66325E-05	9.33784E-09
5	0.000118485	1.40386E-08		9.12689E-05	8.33001E-09
6	6.68116E-05	4.46379E-09		8.99826E-05	8.09686E-09
7	0.000122481	1.50017E-08		9.72267E-05	9.45303E-09
sum	8.553576E-04	1.113997E-07		6.838572E-04	6.725835E-08

(a) Compute the sample mean of the diffusivity for both hydrogen and deuterium.
(b) Compute the sample variance of the diffusivity for both hydrogen and deuterium.
(c) What PDF is appropriate for determining a confidence interval on the difference of means?
(d) Find the lower limit on a 98% confidence interval on the difference of means.
(e) Find the upper limit on a 98% confidence interval on the difference of means.
(f) Translate your conclusions from (d) and (e) into a sentence that a non-statistician can understand.
(g) What PDF is appropriate for determining a confidence interval on the ratio of variances?
(h) Find the lower limit on a 95% confidence interval on the ratio of variances.
(i) Find the upper limit on a 95% confidence interval on the ratio of variances.
(j) Translate your conclusions from (h) and (i) into a sentence that a non-statistician can understand.

Solution:

(a) Compute the sample mean of the diffusivity for both hydrogen and deuterium.

$$D_H = \frac{1}{7}\sum_{i=1}^{7} x_{H,i} = \frac{8.5536 \cdot 10^{-4}}{7} = 1.22 \cdot 10^{-4} \text{ cm}^2/\text{s}$$

$$D_D = \frac{1}{7}\sum_{i=1}^{7} x_{D,i} = \frac{6.8385 \cdot 10^{-4}}{7} = 9.77 \cdot 10^{-5} \text{ cm}^2/\text{s}$$

(b) Compute the sample variance of the diffusivity for both hydrogen and deuterium.

$$s_H^2 = \frac{n\sum_{i=1}^{n} x_{H,i}^2 - \left(\sum_{i=1}^{n} x_{H,i}\right)^2}{n(n-1)} = 1.147 \cdot 10^{-9} \text{ cm}^4/\text{s}^2$$

$$s_D^2 = \frac{n\sum_{i=1}^{n} x_{D,i}^2 - \left(\sum_{i=1}^{n} x_{D,i}\right)^2}{n(n-1)} = 7.495 \cdot 10^{-11} \text{ cm}^4/\text{s}^2$$

(c) What PDF is appropriate for determining a confidence interval on the difference of means?

We must use the t-distribution to determine the confidence interval on the difference of means when the population variance is unknown.

(d) Find the lower limit on a 98% confidence interval on the difference of means.
(e) Find the upper limit on a 98% confidence interval on the difference of means.

$$\alpha = \frac{1 - C.I.}{2} = \frac{1 - 0.98}{2} = 0.01$$

$$v = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\left[\left(\frac{s_1^2}{n_1}\right)^2 \Big/ (n_1 - 1)\right] + \left[\left(\frac{s_2^2}{n_2}\right)^2 \Big/ (n_2 - 1)\right]} \quad \text{if } \sigma_1 \neq \sigma_2$$

$$v = 6.78 \approx 7$$

The limits on the t-distribution are given by

```
>> t = icdf('t',0.01,7)

t = -2.997951566868527
```

and for the upper limit

```
>> t = icdf('t',0.99,7)

t =  2.997951566868527
```

We next insert all of these numbers into the equation for the confidence interval on the difference of means.

$$P\left[\left(\overline{X}_1 - \overline{X}_2\right) + t_\alpha \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} < (\mu_1 - \mu_2) < \left(\overline{X}_1 - \overline{X}_2\right) + t_{1-\alpha} \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}\right] = 1 - 2\alpha$$

$$P\left[\begin{array}{l}\left(1.22\cdot10^{-4}-9.77\cdot10^{-5}\right)+-2.99795\sqrt{\dfrac{1.147\cdot10^{-9}}{7}+\dfrac{7.498\cdot10^{-11}}{7}} \\ \qquad <\left(\mu_1-\mu_2\right)<\left(1.22\cdot10^{-4}-9.77\cdot10^{-5}\right)+2.99795\sqrt{\dfrac{1.147\cdot10^{-9}}{7}+\dfrac{7.498\cdot10^{-11}}{7}}\end{array}\right]=0.98$$

$$P\left[-1.51\cdot10^{-5}<\left(\mu_1-\mu_2\right)<6.41\cdot10^{-5}\right]=0.98$$

(f) Translate your conclusions from (d) and (e) into a sentence that a non-statistician can understand.

We are 98% confidence that the difference between the diffusivity of hydrogen and deuterium lies within the range from $-1.51\cdot10^{-5}$ to $6.41\cdot10^{-5}$ cm^2/s. (It appears that hydrogen diffuses more quickly than deuterium, but we are less than 98% sure, since the range does include a negative difference.)

(g) What PDF is appropriate for determining a confidence interval on the ratio of variances?

We must use the F distribution to determine the confidence interval on the ratio of variances.

(h) Find the lower limit on a 95% confidence interval on the ratio of variances.
(i) Find the upper limit on a 95% confidence interval on the ratio of variances.

$$\alpha=\frac{1-C.I.}{2}=0.025$$
$$v_1=n_1-1=6 \qquad v_2=n_2-1=6$$

The limits on the chi-squared distribution are given by

```
>> flo = icdf('F',.025,6,6)

flo =     0.171828492555022
```

and for the upper limit

```
>> fhi = icdf('F',.975,6,6)

fhi =   5.819756578960774
```

We next insert all of these numbers into the equation for the confidence interval on the ratio of variances.

$$P\left[\frac{S_1^2}{S_2^2}\frac{1}{f_{1-\alpha}(v_1,v_2)}<\frac{\sigma_1^2}{\sigma_2^2}<\frac{S_1^2}{S_2^2}\frac{1}{f_{\alpha}(v_1,v_2)}\right]=1-2\alpha$$

$$P\left[\frac{1.147\cdot10^{-9}}{7.498\cdot10^{-11}}\frac{1}{5.8198}<\frac{\sigma_1^2}{\sigma_2^2}<\frac{1.147\cdot10^{-9}}{7.498\cdot10^{-11}}\frac{1}{0.1718}\right]=0.95$$

$$P\left[2.62899<\frac{\sigma_1^2}{\sigma_2^2}<89.0427\right]=0.95$$

(j) Translate your conclusions from (h) and (i) into a sentence that a non-statistician can understand.

We are 95% sure that the ratio of the variance of the diffusivity of hydrogen to the variance of the diffusivity of deuterium falls within the range from 2.63 to 89.04. (We can state that we are more than 95% confident that the diffusivity of hydrogen has a greater variance than that of deuterium since all values in the interval are greater than one.)

Problem 2. (4 points)

What is the probability that the average diffusivity of hydrogen is greater than the average diffusivity of deuterium.

Solution:

$$\mu_1 - \mu_2 = 0$$

$$T = \frac{\left(\overline{X}_1 - \overline{X}_2\right) - \left(\mu_1 - \mu_2\right)}{\sqrt{\left(\frac{s_1^2}{n_1}\right) + \left(\frac{s_2^2}{n_2}\right)}} = \frac{\left(1.22 \cdot 10^{-4} - 9.77 \cdot 10^{-5}\right) - 0}{\sqrt{\frac{1.147 \cdot 10^{-9}}{7} + \frac{7.498 \cdot 10^{-11}}{7}}} = 1.85$$

$v = 6.78 \approx 7$ from problem 1.

```
>> p = cdf('t',1.85,7)

p =  0.946616452981648
```

$$P\left(\mu_1 - \mu_2 > 0\right) = P\left(t < 1.85\right) = 0.9466$$

Given this data, there is a 94.66% probability that the diffusivity of hydrogen is greater than the diffusivity of deuterium.