Exam I
Administered: Wednesday, September 14, 2016
32 points

For each problem part: 0 points if not attempted or no work shown,
1 point for partial credit, <u>if work is shown</u>,
2 points for correct numerical value of solution

Problem 1. (12 points)

Materials scientists are evaluating the conductivity (S/cm) of a proton exchange membrane composed of a blend of two polymers, Nafion & poly-ethylene glycol (PEG) as a function of PEG content (wt%) at a fixed temperature and relative humidity.

sample	PEG content (wt %)	conductivity (S/cm)	PEG^2 (wt $\%^2$)	conductivity ^{2} (S^2/cm^2)	PEG*conductivity (wt %·S/cm)
1	0.00	0.10	0.00	0.01	0.00
2	0.00	0.09	0.00	0.01	0.00
3	0.25	7.81	0.06	61.00	1.95
4	0.25	6.93	0.06	48.02	1.73
5	0.50	10.00	0.25	100.00	5.00
6	0.50	9.82	0.25	96.43	4.91
7	0.75	0.03	0.56	0.00	0.02
8	0.75	0.04	0.56	0.00	0.03
9	1.00	0.00	1.00	0.00	0.00
10	1.00	0.01	1.00	0.00	0.01
sum	5.0000	34.8300	3.7500	305.4741	13.6575

Based on this data, answer the following questions.

(a) Find the variance of the conductivity.
(b) Find the standard deviation of the conductivity.
(c) Find the covariance of the PEG content and conductivity.
(d) Find the correlation coefficient of the PEG content and conductivity.
(e) Are the PEG content and conductivity independent random variables?
(f) What is the physical relationship between the PEG content and conductivity?

Solution:

(a) Find the variance of the conductivity.

$$\sigma_c^2 = E\left[c^2\right] - E\left[c\right]^2 = \frac{305.4741}{10} - \left(\frac{34.83}{10}\right)^2 = 18.42 \text{ S}^2/\text{cm}^2$$

(b) Find the standard deviation of the conductivity.

$$\sigma_c = \sqrt{\sigma_c^2} = \sqrt{18.42} = 4.29 \text{ S/cm}$$

(c) Find the covariance of the PEG content and conductivity.

$$\sigma_{PEG\cdot c} = E[PEG \cdot c] - E[PEG]E[c] = \frac{13.6575}{10} - \frac{5.0}{10} * \frac{34.83}{10} = -0.37575 \text{ wt \%·S/cm}$$

(d) Find the correlation coefficient of the PEG content and conductivity.

$$\sigma_{PEG}^2 = E[PEG^2] - E[PEG]^2 = \frac{3.75}{10} - \left(\frac{5}{10}\right)^2 = 0.125 \text{ wt \%}^2$$

$$\sigma_{PEG} = \sqrt{\sigma_{PEG}^2} = \sqrt{0.125} = 0.353553 \text{ wt \%}$$

$$\rho_{PEG\cdot c} = \frac{\sigma_{PEG\cdot c}}{\sigma_{PEG}\sigma_c} = \frac{-0.37575}{0.353553 \cdot 4.29} = -0.2477$$

(e) Are the PEG content and conductivity independent random variables?

No, the PEG content and conductivity are not independent random variables because the covariance is not zero.

(f) What is the physical relationship between the PEG content and conductivity?

The negative correlation coefficient indicates that the conductivity decreases as the PEG content increases. However a more careful examination of the data shows that there is a maximum in conductivity at an intermediate value of PEG content.

Problem 2. (10 points)

A study of blended membranes discovered that the PEG and Nafion were not uniformly mixed for a given processing condition. In fact, the PEG content within the membrane is given by the following distribution

$$f(w_{PEG}) = \begin{cases} k\left(w_{PEG} - w_{PEG}^2\right) & for\ 0.0 \le w_{PEG} \le 1.0 \\ 0 & otherwise \end{cases}$$

The conductivity, c, can be approximately related to the local PEG content via the function

$$c(w_{PEG}) = 34.0\left(w_{PEG} - w_{PEG}^2\right)\ S/cm \qquad for\ \ 0.0 \le w_{PEG} \le 1.0$$

(a) Is the PDF continuous or discrete?
(b) Find the value of k that normalizes this PDF.
(c) Find the probability a portion of the membrane contains a PEG content less than 0.3.
(d) Find the probability a portion of the membrane contains a PEG content greater than 0.3.
(e) Find the average conductivity.

Solution:

(a) Is this PDF continuous or discrete?

This PDF is continuous.

(b) Find the value of k that normalizes this PDF.

$$\int_{-\infty}^{\infty} f(w_{PEG})dw_{PEG} = \int_{-\infty}^{\infty} k\left(w_{PEG} - w_{PEG}^2\right)dw_{PEG} = k\int_{0}^{1}\left(w_{PEG} - w_{PEG}^2\right)dw_{PEG}$$

$$= k\left[\frac{w_{PEG}^2}{2} - \frac{w_{PEG}^3}{3}\right]_0^1 = k\left[\frac{1}{2} - \frac{1}{3}\right] = k\left[\frac{1}{6}\right] = 1$$

$$k = 6$$

(c) Find the probability a portion of the membrane contains a PEG content less than 0.3.

$$P\left(w_{PEG} < 0.3\right) = \int_{0}^{0.3} f(w_{PEG})dw_{PEG} == \int_{0}^{0.3} k\left(w_{PEG} - w_{PEG}^2\right)dw_{PEG}$$

$$= k\int_{0}^{0.3}\left(w_{PEG} - w_{PEG}^2\right)dw_{PEG} = k\left[\frac{w_{PEG}^2}{2} - \frac{w_{PEG}^3}{3}\right]_0^{0.3} = k\left[\frac{0.3^2}{2} - \frac{0.3^3}{3}\right]$$

$$= k\left[\frac{3\cdot 0.09}{6} - \frac{2\cdot 0.027}{6}\right] = 6\left[\frac{3\cdot 0.09}{6} - \frac{2\cdot 0.027}{6}\right] = 3\cdot 0.09 - 2\cdot 0.027$$

$$= 0.216$$

(d) Find the probability a portion of the membrane contains a PEG content greater than 0.3.

$$P(w_{PEG} > 0.3) = 1 - P(w_{PEG} < 0.3) = 1 - 0.216 = 0.784$$

(e) Find the average conductivity.

$$\mu_c = \int_{-\infty}^{\infty} c(w_{PEG}) f(w_{PEG}) dw_{PEG} = \int_{-\infty}^{\infty} 34.0(w_{PEG} - w_{PEG}^2)6(w_{PEG} - w_{PEG}^2) dw_{PEG}$$

$$= 204\int_0^1 (w_{PEG} - w_{PEG}^2)(w_{PEG} - w_{PEG}^2) dw_{PEG}$$

$$= 204\int_0^1 (w_{PEG}^2 - 2w_{PEG}^3 + w_{PEG}^4) dw_{PEG}$$

$$= 204\left[\frac{w_{PEG}^3}{3} - \frac{2w_{PEG}^4}{4} + \frac{w_{PEG}^5}{5}\right]_0^1 = 204\left[\frac{1}{3} - \frac{2}{4} + \frac{1}{5}\right] = 204\left[\frac{20}{60} - \frac{30}{60} + \frac{12}{60}\right]$$

$$= \frac{204}{30} = 6.8$$

The average conductivity is 6.8 S/cm.

Problem 3. (10 points)

A study is performed involving fuel cell containing a proton exchange membrane composed of a homopolymer (H) or a blend (B) of two polymers. Each fuel cell is evaluated for a one hundred cold start cycles and is categorized as either (P) or (F).

During this study the following information was collected.

# of fuel cells with a homopolymer membrane that passed the test = 10
# of fuel cells with a homopolymer membrane that failed the test = 3
# of fuel cells with a blend membrane that passed the test = 6
# of fuel cells with a blend membrane that failed the test = 1

Using this information, answer the following questions.

(a) Draw a Venn Diagram of the sample space for this experiment.
(b) What is the probability that a fuel cell in this study contained a homopolymer membrane?
(c) What is the probability that a fuel cell passed the test given that it contained a homopolymer membrane?
(d) What is the probability that a fuel cell passed the test given that it contained a blend membrane?
(e) What is the probability that the membrane contained the blend given that the test failed?

Solution:

H = homopolymer
B = blend
P = pass
F = fail

There are a total of 10+3+6+1 = 20 fuel cells
We are given:

$$P(H \cap P) = \frac{10}{20} = 0.5$$

$$P(H \cap F) = \frac{3}{20} = 0.15$$

$$P(B \cap P) = \frac{6}{20} = 0.3$$

$$P(B \cap F) = \frac{1}{20} = 0.05$$

(a) Draw a Venn Diagram of the sample space for this experiment.

$H \cap P$	$H \cap F$
$B \cap P$	$B \cap F$

(b) What is the probability that a fuel cell in this study contained a homopolymer membrane?

We can use the union rule.

$$P(H) = P(H \cap P) + P(H \cap F) - P[(H \cap P) \cup (H \cap F)]$$

There is no intersection of short and long polishing times.

$$P(A) = 0.5 + 0.15 - 0 = 0.65$$

(c) What is the probability that a fuel cell passed the test given that it contained a homopolymer membrane?

Consider the conditional probability rule.

$$P(P \mid H) = \frac{P(P \cap H)}{P(H)} = \frac{0.5}{0.65} \approx 0.769$$

(d) What is the probability that a fuel cell passed the test given that it contained a blend membrane?

$$P(B) = 1 - P(H) = 1 - 0.65 = 0.35$$

$$P(P \mid B) = \frac{P(P \cap B)}{P(B)} = \frac{0.3}{0.35} \approx 0.857$$

(e) What is the probability that the membrane contained the blend given that the test failed?

$$P(F) = P(H \cap F) + P(B \cap F) - P\left[(H \cap F) \cup (B \cap F)\right]$$

$$P(F) = 0.15 + 0.05 - 0.0 = 0.2$$

$$P(B \mid F) = \frac{P(B \cap F)}{P(F)} = \frac{0.05}{0.2} = 0.25$$