Exam I
Administered: Wednesday, September 16, 2015
32 points

For each problem part: 0 points if not attempted or no work shown,
1 point for partial credit, if work is shown,
2 points for correct numerical value of solution

Problem 1. (12 points)

Materials Scientists students are polishing 8 ceramic samples with an abrasive with a 320 grit. The following table provides the length of time the sample was polished (s) and the RMS surface roughness (μm).

sample	time (s)	surface roughness (μm)	time^2 (s^2)	surface roughness^2 (μm^2)	time*surface roughness (s·μm)
1	60	1.7	3600	2.89	102
2	60	1.8	3600	3.24	108
3	120	0.8	14400	0.64	96
4	120	0.68	14400	0.4624	81.6
5	180	0.46	32400	0.2116	82.8
6	240	0.34	57600	0.1156	81.6
7	300	0.21	90000	0.0441	63
8	600	0.24	360000	0.0576	144
sum	1680	6.23	576000	7.6613	759

Based on this data, answer the following questions.

(a) Find the variance of the surface roughness.
(b) Find the standard deviation of the surface roughness.
(c) Find the covariance of polishing time and surface roughness.
(d) Find the correlation coefficient of polishing time and surface roughness.
(e) Are the polishing time and surface roughness independent random variables?
(f) What is the physical relationship between polishing time and surface roughness based on your answer to (d)?

Solution:

(a) Find the variance of the surface roughness.

$$\sigma_{sr}^2 = E\left[sr^2\right] - E\left[sr\right]^2 = \frac{7.6613}{8} - \left(\frac{6.23}{8}\right)^2 = 0.351 \ \mu\text{m}^2$$

(b) Find the standard deviation of the surface roughness.

$$\sigma_{sr} = \sqrt{\sigma_{sr}^2} = \sqrt{0.351} = 0.593 \ \mu\text{m}$$

(c) Find the covariance of polishing time and surface roughness.

$$\sigma_{t \cdot sr} = E[t \cdot sr] - E[t]E[sr] = \frac{759}{8} - \frac{1680}{8} * \frac{6.23}{8} = -68.66 \text{ s·µm}$$

(d) Find the correlation coefficient of polishing time and surface roughness.

$$\sigma_t^2 = E[t^2] - E[t]^2 = \frac{576000}{8} - \left(\frac{1680}{8}\right)^2 = 27900 \text{ s}^2$$

$$\sigma_t = \sqrt{\sigma_t^2} = \sqrt{27900} = 167.03 \text{ s}$$

$$\rho_{t \cdot sr} = \frac{\sigma_{t \cdot sr}}{\sigma_t \, \sigma_{sr}} = \frac{-68.66}{167.03 \cdot 0.593} = -0.694$$

(e) Are the polishing time and surface roughness independent random variables?

No, the polishing time and surface roughness are not independent random variables because the covariance is not zero.

(f) What is the physical relationship between polishing time and surface roughness based on your answer to (d)?

The negative correlation coefficient indicates that the surface roughness decreases as the polishing time increases, which accords with our intuitive notion of the physical process of polishing.

Problem 2. (10 points)

A study of students in a polishing laboratory discover that, regardless of the recommended polishing time, the amount of time students polish a sample, t, is historically given by the following probability distribution function.

$$f(t) = \begin{cases} c/t & \text{for } 60 \le t \le 600 \\ 0 & \text{otherwise} \end{cases}$$

The surface roughness, s, can be approximately related to the polishing time via the function

$$s(t) = 120/t \ \mu m \qquad \text{for } 60 \le t \le 600$$

(a) Is the PDF continuous or discrete?
(b) Find the value of c that normalizes this PDF.
(c) Find the probability a student polishes a sample less than five minutes.
(d) Find the probability a student polishes a sample more than five minutes.
(e) Find the average surface roughness.

Solution:

(a) Is this PDF continuous or discrete?

This PDF is continuous.

(b) Find the value of c that normalizes this PDF.

$$\int_{-\infty}^{\infty} f(t)dt = \int_{-\infty}^{\infty} \frac{c}{t} dt = c\int_{60}^{600} \frac{1}{t} dt = c[\ln(t)]_{60}^{600} = c[\ln(600) - \ln(60)] =$$
$$= c\ln\left(\frac{600}{60}\right) = c\ln(10) = 1$$

$$c = \frac{1}{\ln(10)} \approx 0.434294482$$

(c) Find the probability a student polishes a sample less than five minutes.

$$P(t < 5\min) = P(t < 300\,s) = \int_{60}^{300} f(t)dt = c\int_{60}^{300} \frac{1}{t} dt = c[\ln(t)]_{60}^{300} = c[\ln(300) - \ln(60)] =$$
$$= c\ln\left(\frac{300}{60}\right) = c\ln(5) = \frac{\ln(5)}{\ln(10)} \approx 0.69897$$

(d) Find the probability a student polishes a sample more than five minutes.

$$P(t > 300\,s) = 1 - P(t < 300\,s) \approx 1 - 0.69897 = 0.30103$$

(e) Find the average surface roughness.

$$\mu_s = \int_{-\infty}^{\infty} s(t) f(t) dt = \int_{-\infty}^{\infty} \frac{120}{t} \frac{c}{t} dt = c \int_{60}^{600} \frac{120}{t^2} dx = -c120 \left[\frac{1}{t} \right]_{60}^{600} = -c120 \left[\frac{1}{600} - \frac{1}{60} \right] =$$

$$= c120 \left[\frac{1}{60} - \frac{1}{600} \right] = c120 \left[\frac{10}{600} - \frac{1}{600} \right] = c120 \frac{9}{600} = c \frac{9}{5} = 0.782 \, \mu m$$

Problem 3. (10 points)

A manufacturer of abrasives intended for polishing is evaluating two competing products, labeled A and B. Product A comes with instructions that say the abrasive should be used for at least five minutes, while product B is to be used for at least 10 minutes. They issued a survey asking users of both products whether they polished for the recommended time. The following information was collected.

\# of respondents who used abrasive A for at least the recommended polishing time = 50
\# of respondents who used abrasive A for less than the recommended polishing time = 20
\# of respondents who used abrasive B for at least the recommended polishing time = 10
\# of respondents who used abrasive B for less than the recommended polishing time = 20

Using this information, answer the following questions.

(a) Draw a Venn Diagram of the sample space for this experiment.
(b) What is the probability that a person bought abrasive A?
(c) What is the probability that a person sufficiently polished a sample given that they bought abrasive A?
(d) What is the probability that a person sufficiently polished a sample given that they bought abrasive B?
(e) What is the probability that a person bought abrasive B given that they sufficiently polished a sample?

Solution:

A = abrasive A
B = abrasive B
L = long polishing time
S = short polishing time
We are given:

$$P(A \cap L) = \frac{50}{100} = 0.5$$
$$P(A \cap S) = \frac{20}{100} = 0.2$$
$$P(B \cap L) = \frac{10}{100} = 0.1$$
$$P(B \cap S) = \frac{20}{100} = 0.2$$

(a) Draw a Venn Diagram of the sample space for this experiment.

$A \cap L$	$A \cap S$
$B \cap L$	$B \cap S$

(b) What is the probability that a person bought abrasive A?

We can use the union rule.

$$P(A) = P(A \cap L) + P(A \cap S) - P[(A \cap L) \cup (A \cap S)]$$

There is no intersection of short and long polishing times.

$$P(A) = 0.5 + 0.2 - 0 = 0.7$$

(c) What is the probability that a person sufficiently polished a sample given that they bought abrasive A?

Consider the conditional probability rule.

$$P(L \mid A) = \frac{P(L \cap A)}{P(A)} = \frac{0.5}{0.7} \approx 0.7143$$

(d) What is the probability that a person sufficiently polished a sample given that they bought abrasive B?

$$P(B) = 1 - P(A) = 1 - 0.7 = 0.3$$

$$P(L \mid B) = \frac{P(L \cap B)}{P(B)} = \frac{0.1}{0.3} \approx 0.33333$$

(e) What is the probability that a person bought abrasive B given that they sufficiently polished a sample?

$$P(L) = P(L \cap A) + P(L \cap B) - P\left[(L \cap A) \cup (L \cap B)\right]$$

$$P(L) = 0.5 + 0.1 = 0.6$$

$$P(B \mid L) = \frac{P(B \cap L)}{P(L)} = \frac{0.1}{0.6} \approx 0.1667$$