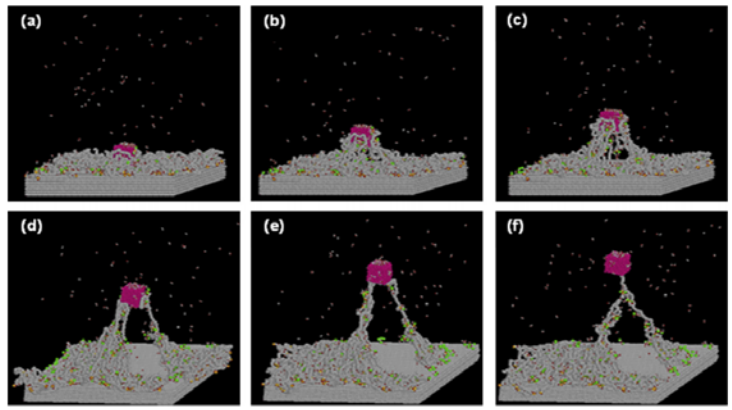
Exam II Solutions
Administered: Friday, October 10, 2014
24 points

For each problem part: 0 points if not attempted or no work shown,
1 point for partial credit, if work is shown,
2 points for correct numerical value of solution

Problem 1. (16 points)

We are evaluating the energy of adhesion between catalyst nanoparticles and a carbon substrate covered in a polymer film. See attached Figure for snapshots of a molecular dynamics simulation in which the Pt nanoparticle (pink) is detached from the surface. We repeat the simulation 8 times. Due to the variable nature of the polymer bridging that forms during the detachment process, there is significant variation in the simulation repetitions. We obtain the following values for the energy of adhesion.

Energy of Adhesion (aJ)
-11.44
-11.04
-10.81
-8.34
-10.24
-10.29
-8.95
-11.27



Snapshots illustrating the process of Pt detachment from the graphite surface coated in a Nafion film for a system includes a 2 nm cubic Pt at the hydration level of $\lambda = 3\ H_2O/SO_3^-$.

From: He, Q., Joy, D.C., Keffer, D.J., "Nanoparticle Adhesion in PEM Fuel Cell Electrodes", J. Power Sources 241 2013 pp. 634-646, doi: 10.1016/j.jpowsour.2013.05.011.

(a) Compute the sample mean.
(b) Compute the sample variance.
(c) What PDF is appropriate for determining a confidence interval on the mean?
(d) Find the lower limit on a 98% confidence interval on the mean.
(e) Find the upper limit on a 98% confidence interval on the mean.
(f) What PDF is appropriate for determining a confidence interval on the variance?
(g) Find the lower limit on a 90% confidence interval on the variance.
(h) Find the upper limit on a 90% confidence interval on the variance.

Solution:

(a) Compute the sample mean.

$$\overline{x} = \frac{1}{8}\sum_{i=1}^{8} x_i = -10.30$$

(b) Compute the sample variance.

$$s^2 = \frac{1}{8-1}\sum_{i=1}^{8}\left[\left(x_i - \overline{x}_2\right)^2\right] = 1.2413$$

(c) What PDF is appropriate for determining a confidence interval on the mean?

We must use the t-distribution to determine the confidence interval on the mean when the population variance is unknown.

(d) Find the lower limit on a 98% confidence interval on the mean.
(e) Find the upper limit on a 98% confidence interval on the mean.

$$v = n - 1 = 7$$
$$s = \sqrt{s^2} = 1.1142$$
$$P(\overline{X} - t_\alpha \frac{s}{\sqrt{n}} < \mu < \overline{X} + t_\alpha \frac{s}{\sqrt{n}}) = 1 - 2\alpha$$

The limits on the t-distribution are given by

```
>> t = icdf('t',0.01,7)

t =  -2.9980
```

and for the upper limit

```
>> t = icdf('t',0.99,7)

t =  2.9980
```

Substituting these values into the formula for the confidence interval yields

$$P(-10.30 - 2.9980\frac{1.1142}{\sqrt{8}} < \mu < -10.30 + 2.9980\frac{1.1142}{\sqrt{8}}) = 0.98$$
$$P(\text{-}11.48 < \mu < \text{-}9.12) = 0.98$$

(f) What PDF is appropriate for determining a confidence interval on the variance?

We must use the chi-squared distribution to determine the confidence interval on the variance.

(g) Find the lower limit on a 90% confidence interval on the variance.
(h) Find the upper limit on a 90% confidence interval on the variance.

$$v = n - 1 = 7$$
$$P\left[\frac{(n-1)s^2}{\chi^2_{1-\alpha}} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{\alpha}}\right] = 1 - 2\alpha$$

The limits on the chi-squared distribution are given by

```
>> chi2 = icdf('chi2',0.05,7)

chi2 =   2.1673
```

and for the upper limit

```
>> chi2 = icdf('chi2',0.95,7)

chi2 =   14.0671
```

Substituting these values into the formula for the confidence interval yields

$$P\left[\frac{(8-1)1.2413}{14.0671} < \sigma^2 < \frac{(8-1)1.2413}{2.1673}\right] = 0.90$$

$$P\left[0.62 < \sigma^2 < 4.01\right] = 0.90$$

Problem 2. (6 points)

A smoke detector is powered by four batteries. Each battery has a mean life time of 36 months. The probability distribution describing the lifetime of a single battery is given by the gamma distribution with parameters α = 36 and β = 1. The smoke detector only operates if all four batteries continue to function.

(a) What is the probability that an individual battery is operating after 24 months?
(b) What PDF would describe the probability that all 4 batteries are functioning after 24 months?
(c) What is the probability that the smoke detector still operates after 24 months?

Solution:

(a) What is the probability that an individual battery is operating after 24 months?

We want the probability that the lifetime is greater than 24.

$$P(x > 24) = \int_{24}^{\infty} f_{\Gamma}(x;\alpha,\beta)dx$$

Matlab provides functions for evaluating the cumulative distribution function,

$$P(x < 24) = \int_{0}^{24} f_{\Gamma}(x;\alpha,\beta)dx$$

```
>> p = cdf('gamma',24,36,1)

p =    0.0132
```

$$P(t > 24) = 0.0132$$

Therefore, the probability that the lifetime is greater than 24 is given by

$$P(x > 24) = 1 - P(t > 24) = 1 - 0.0132 = 0.9868$$

(b) What PDF would describe the probability that all 4 batteries are functioning after 24 months?

This calls for the binomial probability because the batteries are independent and all have the same probability of operating after 24 months have passed.

(c) What is the probability that the smoke detector still operates after 24 months?

x = random variable = number of batteries operating after 24 months = 4
n = total number of batteries = 4
p = probability that an individual battery is operating after 24 months = answer to part (a)

$$P(X = x) = b(x; n, p) = \binom{n}{x} p^x q^{n-x}$$

$$P(x = 4) = b(4;4,0.9868)$$

From Matlab

```
>> p = pdf('binomial',4,4,0.9868)

p =    0.9482
```

$$P(x = 4) = 0.9482$$

The probability that the smoke detector still operates after 24 months is 0.9482.

Problem 2. (8 points) (Exponential Version)

A smoke detector is powered by four batteries. Each battery has a mean life time of 36 months. The smoke detector only operates if all four batteries continue to function.

(a) What PDF would describe the probability that an individual battery is operating after 24 months?
(b) What is the probability that an individual battery is operating after 24 months?
(c) What PDF would describe the probability that all 4 batteries are functioning after 24 months?
(d) What is the probability that the smoke detector still operates after 24 months?

Solution:

(a) What PDF would describe the probability that an individual battery is operating after 24 months?

The exponential PDF would describe the life-time of a single battery.

(b) What is the probability that an individual battery is operating after 24 months?

$$P(t > 24) = \int_{24}^{\infty} f_e(t;\beta)dt = \int_{24}^{\infty} \frac{1}{\beta} e^{-\frac{t}{\beta}} dt = -e^{-\frac{t}{\beta}}\Big|_{24}^{\infty} = -e^{-\frac{\infty}{\beta}} - -e^{-\frac{24}{\beta}} = 0 + e^{-\frac{24}{\beta}}$$

$$P(t > 24) = e^{-\frac{24}{36}} = 0.5134$$

(c) What PDF would describe the probability that all 4 batteries are functioning after 24 months?

This calls for the binomial probability because the batteries are independent and all have the same probability of operating after 24 months have passed.

(d) What is the probability that the smoke detector still operates after 24 months?

x = random variable = number of batteries operating after 24 months = 4
n = total number of batteries = 4
p = probability that an individual battery is operating after 24 months = answer to part (b)

$$P(X = x) = b(x; n, p) = \binom{n}{x} p^x q^{n-x}$$

$$P(x = 4) = b(4;4,0.5134)$$

From Matlab

```
>> p = pdf('binomial',4,4,0.5134)

p =    0.0695
```

$$P(x = 4) = b(4;4,0.5134) = 0.0695$$