Exam I
Administered: Wednesday, September 17, 2014
26 points

For each problem part: 0 points if not attempted or no work shown,
1 point for partial credit, if work is shown,
2 points for correct numerical value of solution

Problem 1. (10 points)

Consider the following mean values obtained for two discrete random variables, x and y.

variable	X	Y	X^2	Y^2	XY
mean	13.13	4.21	180.19	26.19	52.51

(a) Find the variance of X.
(b) Find the standard deviation of X.
(c) Find the covariance of x and y.
(d) Find the correlation coefficient of x and y.
(e) Are x and y independent random variables?

Solution:

(a) Find the variance of X.

$$\sigma_X^2 = E[x^2] - E[x]^2 = 180.19 - 13.13^2 = 7.79$$

(b) Find the standard deviation of X.

$$\sigma_X = \sqrt{\sigma_X^2} = \sqrt{7.79} = 2.79$$

(c) Find the covariance of x and y.

$$\sigma_{XY} = E[xy] - E[x]E[y] = 52.51 - 13.13 * 4.21 = -2.76$$

(d) If the standard deviation of Y is 0.3433, find the correlation coefficient of x and y.

$$\sigma_Y^2 = E[y^2] - E[y]^2 = 26.19 - 4.21^2 = 8.47$$

$$\sigma_Y = \sqrt{\sigma_Y^2} = \sqrt{8.47} = 2.91$$

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \frac{-2.76}{2.79 \cdot 2.91} = -0.34$$

(e) Are x and y independent random variables?

No, x and y are not independent random variables because the covariance is not zero. In fact, x and y are negatively correlated.

Problem 2. (8 points)

Consider the following PDF

$$f(x) = \begin{cases} c\left[(x-1)^2 + 2\right] & \textit{for } -1 \le x \le 1 \\ 0 & \textit{otherwise} \end{cases}$$

(a) Is this PDF continuous or discrete?
(b) Find the value of c that normalizes this PDF.
(c) Find the probability that x is negative.
(d) Find the probability that x is positive

Solution:

(a) Is this PDF continuous or discrete?

This PDF is continuous.

(b) Find the value of c that normalizes this PDF.

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} c\left[(x-1)^2 + 2\right]dx = c\int_{-1}^{1}\left[(x-1)^2 + 2\right]dx = c\left[\frac{(x-1)^3}{3} + 2x\right]_{-1}^{1}$$

$$= c\left[0 + 2 - -\frac{8}{3} - -2\right] = c\frac{20}{3} = 1$$

$$c = \frac{3}{20}$$

(c) Find the probability that x is negative.

$$P(x<0) = \int_{-\infty}^{0} f(x)dx = \int_{-1}^{0} c\left[(x-1)^2 + 2\right]dx = c\left[\frac{(x-1)^3}{3} + 2x\right]_{-1}^{0}$$

$$= c\left[-\frac{1}{3} - -\frac{8}{3} - -2\right] = c\frac{13}{3} = \frac{13}{20}$$

(d) Find the probability that x is positive.

$$P(0<x) = 1 - P(x<0) = 1 - \frac{13}{20} = \frac{7}{20}$$

Problem 3. (10 points)

A recent study has shown positive statistical correlations between wealth and unethical behavior.* One study examined whether drivers of fancy (F) or junky (J) cars obeyed (O) the law and yielded to pedestrians in a crosswalk or disobeyed (D) the law. In this similar problem, the scientists observed 150 cars for this study. Answer the questions below based on the following results. This table provides the number of drivers.

	driving fancy car (F)	driving junky car (J)
obeyed the law (O)	30	50
disobeyed the law (D)	30	40

*Piffa, P.K, Stancatoa, D.M., Côtéb, S., Mendoza-Dentona, R. Keltnera, D., “Higher social class predicts increased unethical behavior”, Proceedings of the National Academy of Sciences of the United States of America, vol. 109(11), March 13, 2012, pp.4086–4091.

Using this information, answer the following questions.

(a) Draw a Venn Diagram of the sample space for this experiment.
(b) What is the probability that a person drives a junky car?
(c) What is the probability that a person disobeyed the law given that they drive a junky car?
(d) What is the probability that a person disobeyed the law given that they drive a fancy car?
(e) What is the probability that a person drove a fancy car given that they disobeyed the law?
(f) Comparison of which of these two results supports the conclusion that people who drive fancy cars are more likely to disobey this law.

Solution:

We are given:

$$P(O \cap F) = \frac{30}{150} = 0.2$$
$$P(O \cap J) = \frac{50}{150} = 0.333$$
$$P(D \cap F) = \frac{30}{150} = 0.2$$
$$P(D \cap J) = \frac{40}{150} = 0.267$$

(a) Draw a Venn Diagram of the sample space for this experiment.

$O \cap F$	$O \cap J$
$D \cap F$	$D \cap J$

(b) What is the probability that a person drives a junky car?

We can use the union rule.

$$P(J) = P(O \cap J) + P(D \cap J) - P\left[(O \cap J) \cup (D \cap J)\right]$$

There is no intersection of obeying and disobeying.

$$P(J) = 0.333 + 0.267 - 0 = 0.6$$

(c) What is the probability that a person disobeyed the law given that they drive a junky car?

Consider the conditional probability rule.

$$P(D \mid J) = \frac{P(D \cap J)}{P(J)} = \frac{0.267}{0.6} = 0.444$$

(d) What is the probability that a person disobeyed the law given that they drive a fancy car?

$$P(F) = 1 - P(J) = 1 - 0.6 = 0.4$$

$$P(D \mid F) = \frac{P(D \cap F)}{P(F)} = \frac{0.2}{0.4} = 0.5$$

(e) What is the probability that a person drove a fancy car given that they disobeyed the law?

$$P(D) = P(D \cap F) + P(D \cap J) - P\left[(D \cap F) \cup (D \cap J)\right]$$

$$P(D) = 0.2 + 0.267 = 0.467$$

$$P(F \mid D) = \frac{P(D \cap F)}{P(D)} = \frac{0.2}{0.467} = 0.429$$

(f) Comparison of which of these two results supports the conclusion that people who drive fancy cars are more likely to disobey this law.

Comparing (c) and (d) yield this conclusion because (c) says that 44% of drivers in junky cars broke the law but 50% of drivers in fancy cars broke the law.