Midterm Examination Number Two
Administered: Wednesday, October 6, 2004

In all relevant problems: WRITE DOWN THE FORMULA YOU USE, BEFORE YOU USE IT.

Problem 1.

We are in the business of manufacturing injection-molded plastic fenders to automobile makers. We claim that our fenders will remain intact under head-on impact with a standard concrete pylon up to an average speed of 25 mph with a standard deviation of 3 mph. Our competitor claims that they have developed a new additive to their plastic which allows their bumpers to remain intact under the same conditions up to an average speed of 30 mph with a standard deviation of 5 mph. We want to test their claim. We test 12 fenders, half with our fenders and half with the competition's fenders. From this sample, we find that our bumpers do not fracture until 25.2 mph with a standard deviation of 2.9 mph. From the competition's sample, we find that their bumpers do not fracture until 30.1 mph with a standard deviation of 10 mph.

(a) Find a 98% confidence interval for the difference in the two companies' product's average life-times, assuming the claimed population standard deviations are believable.

(b) Does the claimed average life-time difference fall within this confidence interval?

(c) Find a 98% confidence interval for the difference in the two companies' product's average life-times, assuming the claimed population standard deviations are not believable.

(d) Does the claimed average life-time difference fall within this confidence interval?

(e) Does the test in (c) allow for the possibility that our product has a higher mean than the competition?

Solution:

$\mu_1 = 25$, $\sigma_1 = 3$, $\sigma_1^{\ 2} = 9$, $n_1 = 6$, $\bar{x}_1 = 25.2$, $s_1 = 2.9$, $s_1^{\ 2} = 8.41$

$\mu_2 = 30$, $\sigma_2 = 5$, $\sigma_2^{\ 2} = 25$, $n_2 = 6$, $\bar{x}_2 = 30.1$, $s_2 = 10$, $s_2^{\ 2} = 100$

(a) Find a 98% confidence interval for the difference in the two companies' product's average life-times, assuming the claimed population standard deviations are believable.

We assume that the population standard deviations are given, so we use the z-distribution.

$1 - 2\alpha = 0.98$, $\alpha = 0.01$, $z_\alpha = z_{0.01} = -2.33$, $z_{1-\alpha} = -z_\alpha = 2.33$

$$P\left[\left(\bar{X}_1 - \bar{X}_2\right) + z_\alpha\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} < \left(\mu_1 - \mu_2\right) < \left(\bar{X}_1 - \bar{X}_2\right) - z_\alpha\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}\right] = 1 - 2\alpha$$

$$P\left[\left(25.2 - 30.1\right) - 2.33\sqrt{\frac{9}{6} + \frac{25}{6}} < \left(\mu_1 - \mu_2\right) < \left(25.2 - 30.1\right) + 2.33\sqrt{\frac{9}{6} + \frac{25}{6}}\right] = 0.98$$

$$P\left[-10.45 < \left(\mu_1 - \mu_2\right) < 0.65\right] = 0.98$$

so the 98% confidence interval for the mean is

$$-10.45 < \left(\mu_1 - \mu_2\right) < 0.65$$

(b) Does the claimed average life-time difference fall within this confidence interval?

Yes, the claimed difference of population means $\mu_1 - \mu_2 = -5$ falls within this interval.

(c) Find a 98% confidence interval for the difference in the two companies' product's average life-times, assuming the claimed population standard deviations are not believable.

We assume that the population standard deviations are not given, so we use the t-distribution.

$1-2\alpha = 0.98$, $\alpha = 0.01$, $t_\alpha = t_{0.01} = -2.33$, $t_{1-\alpha} = -t_\alpha = 2.33$

$$v = \frac{\left(\frac{s_1^2}{n_1}+\frac{s_2^2}{n_2}\right)^2}{\left[\left(\frac{s_1^2}{n_1}\right)^2 \Big/ (n_1-1)\right]+\left[\left(\frac{s_2^2}{n_2}\right)^2 \Big/ (n_2-1)\right]} \text{ if } \sigma_1 \neq \sigma_2$$

$$v = \frac{\left(\frac{8.41}{6}+\frac{100}{6}\right)^2}{\left[\left(\frac{8.41}{6}\right)^2 \Big/ (6-1)\right]+\left[\left(\frac{100}{6}\right)^2 \Big/ (6-1)\right]} = 5.8351 \approx 6$$

$t_\alpha = t_{0.01} = -3.143$, $t_{1-\alpha} = -t_\alpha = 3.143$

The t value came from the Table A.4, of the t-PDF values.

$$P\left[(25.2-30.1)-3.143\sqrt{\frac{8.41}{6}+\frac{100}{6}} < (\mu_1-\mu_2) < (25.2-30.1)+3.143\sqrt{\frac{8.41}{6}+\frac{100}{6}}\right] = 0.98$$

$$P[-18.2599 < (\mu_1-\mu_2) < 8.4599] = 0.98$$

so the 98% confidence interval for the mean is

$$-18.3 < (\mu_1-\mu_2) < 8.5$$

(d) Does the claimed average life-time difference fall within this confidence interval?

Yes, the claimed difference of population means $\mu_1 - \mu_2 = -5$ falls within this interval.

(e) Does the test in (c) allow for the possibility that our product has a higher mean than the competition?

Yes, it does because positive values of $\mu_1 - \mu_2$ are included in the confidence interval.

Problem 2.

In working out a multiple choice test with choices A, B, C & D, we run out of time with 10 questions left. We fill all the remaining questions out with C. If the probability that the correct answer is A, B, C or D is the same and is independent of all other problems, find the following probablilities.

(a) What is the probability that we got 5 of the last 10 questions right?

(b) What is the probability that we got more than 8 of the last 10 questions right?

Solution:

Use binomial PDF because the problems are independent, and you have two choices (getting the question right or wrong), and the probability doesn't change from one problem to the next.

x = number of correct answers   n = 10   p = 0.25

(a) What is the probability that we got 5 of the last 10 questions right?

$$P(x=5)=b(5;10,0.25)=\frac{10!}{5!5!}0.25^5 0.75^5=0.05840$$

(b) What is the probability that we got more than 8 of the last 10 questions right?

$$P(x>8)=P(x=9)+P(x=10)=b(9;10,0.25)+b(10;10,0.25)$$

$$P(x>8)=\frac{10!}{9!1!}0.25^9 0.75^1+\frac{10!}{10!0!}0.25^{10}0.75^0=2.861\cdot10^{-5}+9.537\cdot10^{-7}=2.956\cdot10^{-5}$$

Problem 3.

Driving to school each morning, we encounter 5 traffic lights. Each traffic light stays green for 45 seconds, yellow for 5 seconds, and red for 50 seconds.
(a) Assuming that there is absolutely no synchronization among the streetlights and assuming that we don't run yellow lights, find the probability that in a single morning, we hit 2 green lights, 2 yellow lights, and 1 red lights.
(b) What is the average time spent waiting on lights on a single morning? Assume your arrival time is uniformly distributed among the lights.

Solution.

Use the multinomial PDF.

$$P(\{X=x\})=m(\{x\};n,\{p\},k)=\binom{n}{x_1,x_2\ldots x_k}\prod_{i=1}^{k}p_i^{x_i}$$

$$P(\{X=x\})=\binom{5}{2,2,1}(0.45)^2(0.05)^2(0.50)^1$$

$$P(\{X=x\})=\frac{5!}{2!2!1!}(0.45)^2(0.05)^2(0.50)^1=0.007594$$

(b) What is the average time spent waiting on lights on a single morning? Assume your arrival time is uniformly distributed among the lights.

Definition of the average time.

$$\mu_{g(x)}=E(g(x))=\sum_x g(x)f(x)$$

Definition of the PDF.

$$f(x) = \begin{cases} 0.45 & green \\ 0.05 & yellow \\ 0.50 & red \end{cases}$$

g(x) is a function of the random variable representing waiting at the light.
When we have a green light, g(x) is 0 because we don't wait at a green light.
When we have a red light g(x) is something between 0 and 50 seconds. Using the continuous uniform PDF for our arrival time, the average wait at a red light is 50/2 = 25.
When we hit a yellow light, we have to wait between 0 and 5 seconds on the yellow then we have to wait all of the red light, 5/2 + 50 = 52.5.

$$x = \begin{cases} 0 & \text{seconds waiting on } green \\ 52.5 & \text{seconds waiting on } yellow\ then \text{ red} \\ 25 & \text{seconds waiting on } red \end{cases}$$

Plugging f(x) and g(x) into the formula for the average, we have:

$$\mu = 0 \cdot 0.45 + (52.5) \cdot 0.05 + 25 \cdot 0.50 = 15.125 \text{ seconds}$$ per light.

On a given morning, we have 5 lights, so our total waiting time is 76 seconds.