Final Exam
Administered: 10:15 am -12:15 pm, Thursday, December 4, 2003
48 points

For each problem part: 0 points if not attempted or no work shown,
1 point for partial credit, if work is shown,
2 points for correct numerical value of solution

Problem (1) (10 points)
Perform one complete Newton-Raphson iteration on the system of equations:

$$x - 5\exp(y) = -9 \qquad x^3 y = 4$$

Use (x,y) = (1,0) as your initial guess.
Along the way, present the Jacobian (2 points), Residual (2 points), determinant (2 points), inverse of the Jacobian (2 points), and new estimate of [x,y] (2 points).

solution:

$$f_1(x,y) = x - 5\exp(y) + 9 = 0 \qquad f_2(x,y) = x^3 y - 4 = 0$$

$$\underline{\underline{J}} = \begin{bmatrix} 1 & -5\exp(y) \\ 3x^2 y & x^3 \end{bmatrix} \qquad \underline{R} = \begin{bmatrix} x - 5\exp(y) + 9 \\ x^3 y - 4 \end{bmatrix}$$

$$\underline{\underline{J}}(x=1, y=0) = \begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix} \qquad \underline{R}(x=1, y=0) = \begin{bmatrix} x - 5\exp(y) + 9 \\ x^3 y - 4 \end{bmatrix} = \begin{bmatrix} 1 - 5 + 9 \\ 0 - 4 \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$$

$$\det(J) = j_{11}j_{22} - j_{21}j_{12} = (1)(1) - (0)(-5) = 1$$

$$\underline{\underline{J}}^{-1} = \frac{1}{\det(J)}\begin{bmatrix} j_{22} & -j_{12} \\ -j_{21} & j_{11} \end{bmatrix} = \frac{1}{1}\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$$

$$\underline{\delta x} = -\underline{\underline{J}}^{-1}\underline{R} = -\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ -4 \end{bmatrix} = \begin{bmatrix} 15 \\ 4 \end{bmatrix}$$

$$\underline{x}^{(1)} = \underline{x}^{(0)} + \underline{\delta x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 15 \\ 4 \end{bmatrix} = \begin{bmatrix} 16 \\ 4 \end{bmatrix}$$

Problem 2. (4 points)

An outdoor motion sensor is advertised as detecting 90% of human trespassers. During a demonstration test, 20 people walk past the motion sensor.

(a) What is the probability that the motion sensor detects 15 or fewer people?

(b) What is the probability that the motion sensor MISSES for the fifth time on the 20th pass?

Solution:

(a) Binomial PDF.

$$b(x;n,p) = \binom{n}{x} p^x q^{n-x}$$

$p = 0.9$ and $q = 1 - p = 0.1$ and $n = 20$ and $x \le 15$

$P(X \le 15) = B(15;20,0.9) = 0.0432$ From Table A.1

(b) negative binomial PDF.

$$b^*(x;k,p) = \binom{x-1}{k-1} p^k q^{x\text{-}k} \quad \text{for } x = k, k+1, k+2...$$

$p = 0.1$ and $q = 1 - p = 0.9$ and $x = 20$ and $k = 5$

$$P(X = 20) = \binom{20-1}{5-1} (0.1)^5 (0.9)^{20-5} = 0.0080$$

Problem 3. (8 points)

A manufacturer of windshield wipers claims that her product continues to work for 15 months before requiring replacement with a standard deviation of 2 months. ($\mu = 15, \sigma = 2$) You and 12 of your friends all buy these windshield wipers and put them on your automobiles at the same time. You record the time when the windshield wipers must be replaced and find the sample mean to be $\overline{x} = 13$ months and sample standard deviation $s = 1$ month, find a 95% confidence interval for the population mean, assuming the stated population variance is doubtful and not to be trusted. Do your findings validate the manufacturer's claims?

Solution:

To estimate the mean, variance unknown, use the t-distribution.

$v = n - 1 = 12$. First, find $t_{\alpha/2}$ for $\alpha = 0.05$ from table A.4, $t_{0.025} = 2.179$

$$P(\overline{X} - t_{\alpha/2}\frac{s}{\sqrt{n}} < \mu < \overline{X} + t_{\alpha/2}\frac{s}{\sqrt{n}}) = 1 - \alpha$$

$$P(13 - (2.179)\frac{1}{\sqrt{13}} < \mu < 13 + (2.179)\frac{1}{\sqrt{13}}) = 0.95$$

$$P(12.40 < \mu < 13.60) = 0.95$$

Our findings do not validate the manufacturer's claims because their mean does not fall within our confidence interval.

Problem 4b. (10 points)

Consider an nxn matrix, $\underline{\underline{J}}$, with determinant $\det(\underline{\underline{J}}) = -4$. Which of the following statements are true?

(a) The inverse of $\underline{\underline{J}}$ does not exist.

(b) The rows of $\underline{\underline{J}}$ are all linearly independent.

(c) The rank of $\underline{\underline{J}}$ is n.

(d) There is a unique solution to the system of linear equations $\underline{\underline{J}}\underline{x} = \underline{R}$ for any real nx1 vector, $\underline{R}$.

(e) The reduced row echelon form of $\underline{\underline{J}}$ will have at least one row completely filled with zeroes.

Solution:

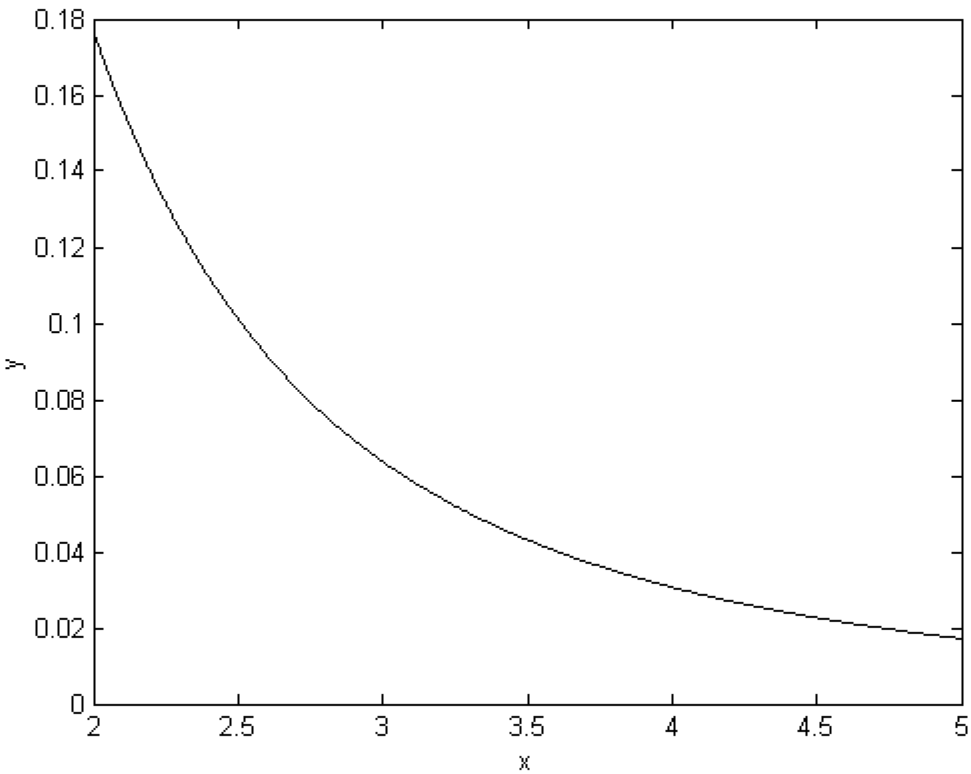
TRUE: (b), (c), (d)

Problem (5). (16 points)

Consider the nonlinear ordinary differential equation initial value problem

$$\frac{dy}{dx} = -\frac{5}{2\left(\sqrt{x}\right)^7} - \frac{\sqrt{y}}{1000}$$

subject to the initial condition $y(x_o = 2) = y_o = 0.17678$ has a solution given in the plot below.



(a) Solve the ODE up to x = 3 using the Euler Method. Use a step size of Δx=0.5
(b) Solve the ODE up to x = 4 using the Euler Method. Use a step size of Δx=1.0
(c) Compare the results of the two different step sizes.

Solution:

(a) Solve the ODE up to x = 3 using the Euler Method. Use a step size of Δx=0.5

The Euler method is given by the equation

$$y_{i+1} = y_i + \Delta x \left(\frac{dy}{dx}\right)_{x=x_i}$$

Applying this formula twice yields

i	x	y(x)	dy/dx
0	2	0.1768	-0.22139131732842
1	2.5	6.608104e-002	-0.10144994744591
2	3	1.535606e-002	

Therefore, y(x=4) =0.015356.

(b) Solve the ODE up to x = 4 using the Euler Method. Use a step size of Δx=1.0

Applying this formula twice yields

i	x	y(x)	dy/dx
0	2	0.1768	-0.22139131732842
1	3	-4.461462e-002	-0.05345835825830 - 0.00021122173665*i* (not a real number)
2	4	(not a real number)	

(c) Compare the results of the two different step sizes.

Therefore, a real value of y(x=4) does not exist. When we take time steps that are too large, our approximation to the real solution is so bad that we obtain negative values of y. Since the derivative is not real for negative values of y, our solution method fails completely.