Derivation of a Numerical Method for solving a system of non-linear parabolic PDEs

I. FORMULATION.

Linear parabolic partial differential equations are, in their most general form, given by:

$$d\frac{\partial T}{\partial t} = \nabla \cdot [c(\nabla T)] - aT - \underline{b} \cdot \nabla T + f \tag{I.1}$$

where the functions, $a, \underline{b}, c, d, f$ are known functions of t, x, y, z and the temperature, T, is an unknown function of t, x, y, z. Frequently, we have not just a single unknown function, T, but a whole family of unknown functions $\left\{ T^{(\ell)} \right\}$ where $1 \le \ell \le n_u$ and n_u is the number of unknown functions. The general form of equation (I.1) for a system of equations is then

$$d^{(\ell)} \frac{\partial T^{(\ell)}}{\partial t} = \sum_{k=1}^{n_u} \left[\nabla \cdot \left[c^{(\ell,k)} \left(\nabla T^{(k)} \right) \right] - a^{(\ell,k)} T^{(k)} - \underline{b}^{(\ell,k)} \cdot \nabla T^{(k)} + f^{(\ell,k)} \right] \tag{I.2}$$

If we have only one spatial dimension, we have

$$d^{(\ell)} \frac{\partial T^{(\ell)}}{\partial t} = \sum_{k=1}^{n_u} \left[c^{(\ell,k)} \frac{\partial^2 T^{(k)}}{\partial x^2} - a^{(\ell,k)} T^{(k)} + \left(\frac{\partial c^{(\ell,k)}}{\partial x} - b_x^{(\ell,k)} \right) \cdot \frac{\partial T^{(k)}}{\partial x} + f^{(\ell,k)} \right] \tag{I.3}$$

We now need to know the functional forms of $a, \underline{b}, c, d, f, \nabla c$, which must be given. All of these functions are now functional matrices of dimension $n_u \mathrm{x} n_u$. In many problems, most of these functions are constants and, often the constants are unity or zero. However, in order to write a code that solves any parabolic PDE, it is for this general formulation that we derive a finite-difference method.

The formulation in equation (I.3) assumes a linear form of the system of parabolic PDEs. The PDEs may not be linear and then our general form is

$$\frac{\partial T^{(\ell)}}{\partial t} = K^{(\ell)}\left(x, t, \{T\}, \{\nabla T\}, \{\nabla^2 T\} \right) \tag{I.4}$$

From this point on, we proceed in exactly the same manner as we did for a single parabolic PDE. If you recall, we developed algorithms for solving the typical linear parabolic PDEs (the heat equation and the convection-diffusion equation) and then we developed a different algorithm for solving a completely general non-linear parabolic PDE. For systems of equations, we certainly could repeat both the linear and non-linear derivations. If we had an infinite amount of time in our lives, we might just do that. Given the finite time constraints of life in this universe, we opt to derive only the algorithm for the solution of a system of general non-linear parabolic PDEs. Of course, we can use this non-linear algorithm to solve linear problems.

A comment on notation: we will write $T^{(\ell)}(t_j, x_i)$ as $T^{(\ell)}{}_i^j$ so that

j superscripts designate temporal increments
i subscripts designate spatial increments
ℓ and k superscripts inside parentheses designate different functions

To recap what we did in the single parabolic PDE case, we first discretized time and space. Second we used the second order Classical Runge-Kutta method to solve the time component of the PDE like an ODE.

$$T^{(\ell)}{}_i^{j+1} = T^{(\ell)}{}_i^j + \frac{\Delta t}{2}\left[K^{(\ell)}{}_i^{j+1} + K^{(\ell)}{}_i^j\right] \tag{I.5}$$

where $K^{(\ell)}{}_i^j$ is the time derivative of $T^{(\ell)}{}_i^j$,

$$K^{(\ell)}{}_i^j = K^{(\ell)}\left(x_i, t_j, \left\{T^j\right\}, \left\{\nabla T^j\right\}, \left\{\nabla^2 T^j\right\}\right) \tag{I.6}$$

The braces in equation (I.6) now stand for the complete set over both position i and function (ℓ).

The second function in equation (I.5) is given by $K(x_i, t_{j+1}, T_i^j + \Delta t K_i^j)$

$$K^{(\ell)}{}_i^{j+1} = K^{(\ell)}\left(x_i, t_{j+1}, \left\{T^j\right\}, \left\{\nabla T^j\right\}, \left\{\nabla^2 T^j\right\}\right) \tag{I.6}$$

where the temperature is given by

$$T_i^{j+1} \approx T_i^j + \Delta t K_i^j \tag{I.7}$$

Note: this temperature is used not only for the explicit temperatures but is also used in the finite difference formulae to obtain the first and second spatial partial derivatives.

We saw that solving a single non-linear parabolic PDE with the algorithm was equivalent to solving a system of m ODEs. We now see that solving a system of n_u non-linear parabolic PDEs is equivalent to solving a system of $m \cdot n_u$ ODEs. Looking at it this way, we don't require any new programs or theories to solve a system of non-linear parabolic PDEs, if we have the tools to solve a single non-linear parabolic PDE.

The boundary conditions are handled the same way as in the single non-linear parabolic PDE algorithm.