Homework Assignment Number Four Solutions
Assigned: Thursday, October 7, 1999
Due: Tuesday, October 19, 1999 BEGINNING OF CLASS.

Problem (1)

Consider the initial value problem:

$$\frac{dy}{dx} + a(x)y = b(x)$$

where we have an initial condition of the form:

$$y(x = x_o) = y_o$$

with the specific values given by:

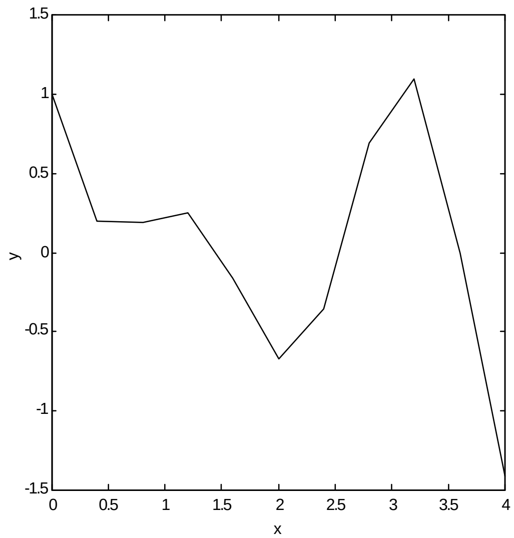
$$a(x) = 2\,,\; b(x) = x\sin(3x)\,,\; y(x = 0) = 1$$

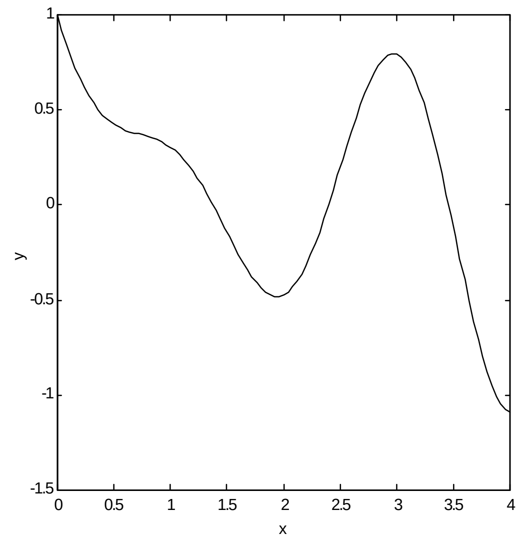
In homework assignment three, you analytically solved and plotted the solution from x = 0 to 4. Now numerical solve same equation.

(a) Use Euler with a time step of 0.4
(b) Use Euler with a time step of 0.04
(c) Use Runge-Kutta with a time step of 0.4
(d) Use Runge-Kutta with a time step of 0.04
(e) Compare the relative error of the Euler estimate of y(x=4) for both sized steps. Explain.
(f) Compare the relative error of the Runge-Kutta estimate of y(x=4) for both sized steps. Explain.
(g) Compare the relative errors of the Euler and Runge-Kutta estimates of y(x=4) for a time step of 0.04. Explain.
(h) Compare the relative errors of the Runge-Kutta estimates of y(x=2) and y(x=4) for a time step of 0.04. Explain.

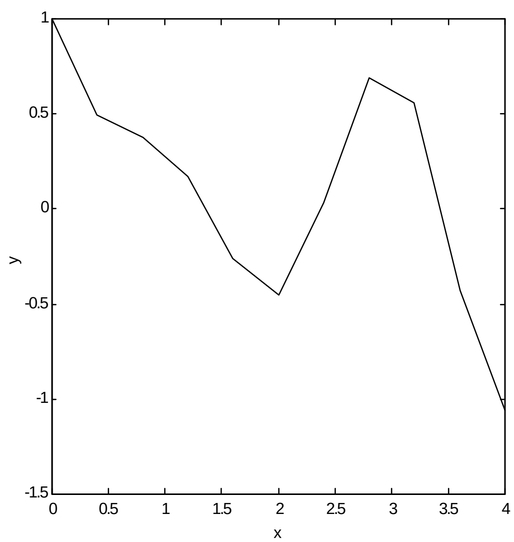
Solution:

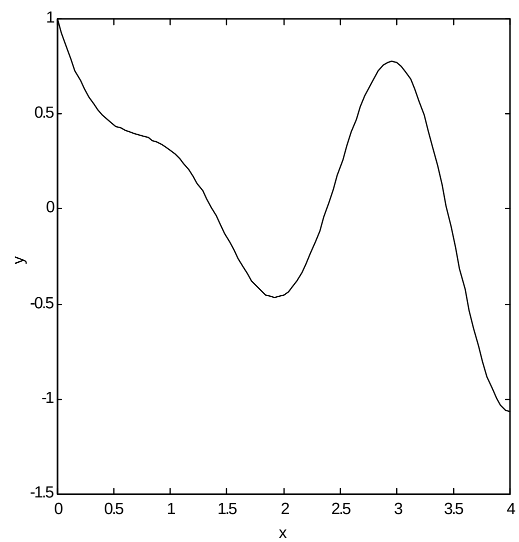
(a) Use Euler with a time step of 0.4
(b) Use Euler with a time step of 0.04





(c) Use Runge-Kutta with a time step of 0.4
(d) Use Runge-Kutta with a time step of 0.04





(e) Compare the relative error of the Euler estimate of y(x=4) for both sized steps. Explain.

analytical solution: y(x=4) = -1.0648
for time step of 0.4, y(x=4) = -1.4134 percent error 32.74%
for time step of 0.04, y(x=4) = -1.0924 percent error 2.59%

since for the Euler method, $\mathrm{error} \propto \Delta t$ then $\frac{\mathrm{error}_1}{\Delta t_1} = \frac{\mathrm{error}_2}{\Delta t_2}$, so that $\frac{\mathrm{error}_1}{\Delta t_1}\frac{\Delta t_2}{\mathrm{error}_2} \approx 1$

Here we find: $\frac{0.3274}{0.4}\frac{0.04}{0.0259} = 1.2624 \approx 1$

(f) Compare the relative error of the Runge-Kutta estimate of y(x=4) for both sized steps. Explain.

analytical solution: y(x=4) = -1.0647855058
for time step of 0.4, y(x=4) = -1.0617 percent error 0.29%
for time step of 0.04, y(x=4) = -1.0647851095e+000 percent error 3.7225e-005%

Since for the Euler method, $\text{error} \propto \Delta t^4$ then $\frac{\text{error}_1}{\Delta t_1^{\,4}} = \frac{\text{error}_2}{\Delta t_2^{\,4}}$, so that $\frac{\text{error}_1}{\Delta t_1}\frac{\Delta t_2}{\text{error}_2} \approx 1$

Here we find: $\frac{0.0029}{0.4^4}\frac{0.04^4}{0.00000037225} = 0.7790 \approx 1$

(g) Compare the relative errors of the Euler and Runge-Kutta estimates of y(x=4) for a time step of 0.04. Explain.

analytical solution: y(x=4) = -1.0648
Euler for time step of 0.04, y(x=4) = -1.0924 percent error 2.59%
RK4 for time step of 0.04, y(x=4) = -1.0648 percent error 0.0013612%

The Runge-Kutta method is much more accurate because it is a fourth order method while the Euler method is only first order.

(h) Compare the relative errors of the Runge-Kutta estimates of y(x=2) and y(x=4) for a time step of 0.04. Explain.

analytical solution: y(x=2) = -0.45220335509
analytical solution: y(x=4) = -1.0647855058
for time step of 0.04, y(x=2) = -0.45220311856 percent error 5.2307e-005%
for time step of 0.04, y(x=4) = -1.0647851095 percent error 3.7225e-005%

The errors appear to be about the same at both points.

Problem (2)

In Homework assignment 3, you plotted the solution to the acetylene vibrational/translational problem for the initial conditions:

$$\underline{x}(t=0) = \begin{bmatrix} -0.1 \\ 0 \\ 0.25 \\ 0.5 \end{bmatrix} \text{ and initial velocities } \dot{\underline{x}}(t=0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

for t = 0 to 2 sec, using the following values for the masses and the spring constants.

$m_H = 1.0$, $m_C = 12.0$, $k_{HC} = 1.0$, $k_{CC} = 10.0$

(a) Solve the problem using the Runge-Kutta fourth-order method.

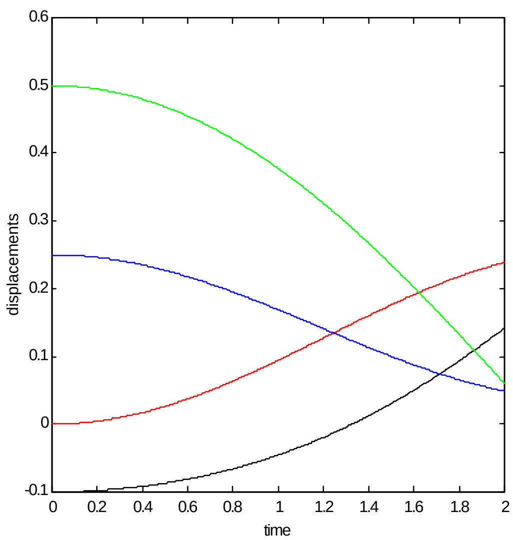
Solved using sysode.m with the following input file:

```
function dydt = sysodeinput(x,y,nvec);
%
%  This is the HCCH problem
%
% sample input:
% sysode(2,100,0,2,[-0.1 0 0.25 0.5 0.0 0.0 0.0 0.0]')
%
mh = 1;
mc = 12;
khc = 1;
kcc = 10;
A(1,:) =[0 0 0 0 1 0 0 0];
A(2,:) =[0 0 0 0 0 1 0 0];
A(3,:) =[0 0 0 0 0 0 1 0];
A(4,:) =[0 0 0 0 0 0 0 1];
A(5,:) =[-khc/mh khc/mh 0 0 0 0 0 0];
A(6,:) =[-khc/mc -(khc+kcc)/mc kcc/mc 0 0 0 0 0];
A(7,:) =[0 kcc/mc -(khc+kcc)/mc khc/mc 0 0 0 0];
A(8,:) =[0 0 khc/mh -khc/mh 0 0 0 0];
dydt = A*y';
```

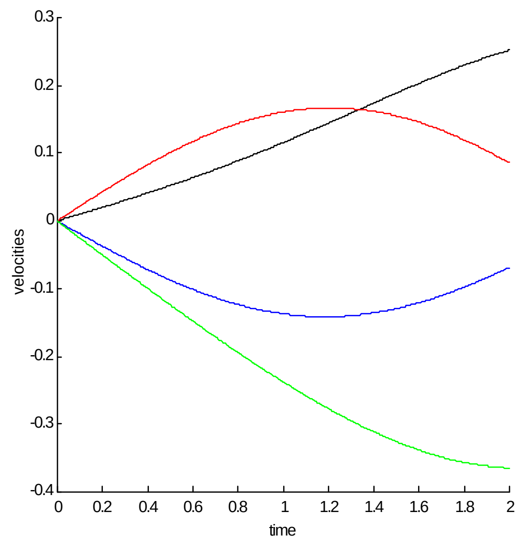
(b) Compare the analytical and numerical solutions.

Solution:

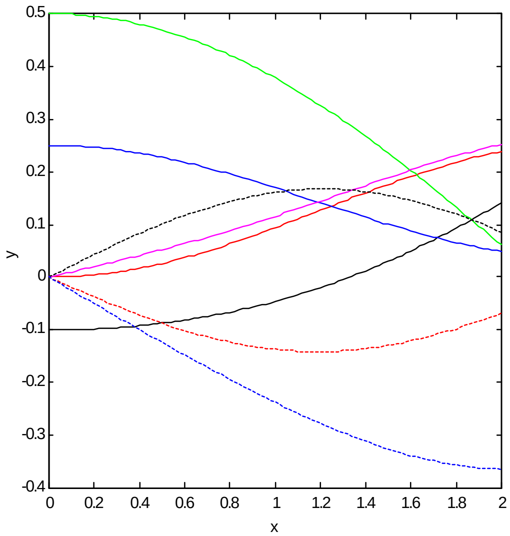
plot of analytical displacements



plot of analytical velocities



numerical results - displacements and velocities



The analytical and numerical solutions are virtually identical up to 2 seconds.

Problem (3)

Consider:

$$\frac{d^2y}{dx^2} = c_1 \frac{dy}{dx} + c_2 y + c_3 \sin(x) + c_4$$

with the boundary conditions

$$y(x = 0) = y_o = 1.0$$
$$y(x = 10) = y_f = 1.0$$

where $c = [1.0, -2.0, 2.0, 0.0,]$

(a) Convert this single second-order ODE, to a system of two first-order ODEs.
(b) Determine the behavior of $y(x)$ and $y'(x)$ from $0 \le x \le 10$. Show the behavior in a plot. Clearly identify which curve corresponds to which function.
(c) State what value of an initial guess for $y'(x = 0)$ led to the final solution.

Solution:

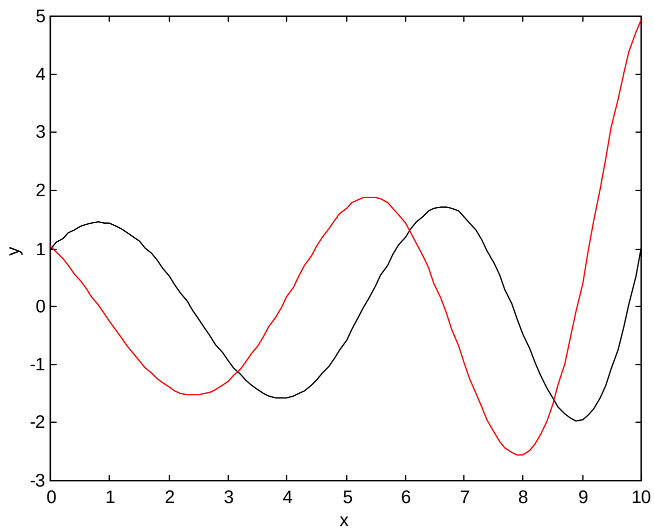
(a) Convert this single second-order ODE, to a system of two first-order ODEs.

$$\frac{dy_1}{dx} = y_2$$

$$\frac{dy_2}{dx} = c_1 y_2 + c_2 y_1 + c_3 \sin(x) + c_4$$

(b) Determine the behavior of $y(x)$ and $y'(x)$ from $0 \le x \le 10$. Show the behavior in a plot. Clearly identify which curve corresponds to which function.

```
» shooting(2,100,0,10,[1,1],1)
iteration    1:  dy/dt guess: 1.0000000e+000, relative error = 2.3838459e+000
iteration    2:  dy/dt guess: -1.0000000e+000, relative error = 1.4037414e+002
iteration    3:  dy/dt guess: 1.0345509e+000, relative error = 2.3092639e-014
```



black curve is y and red curve is dy/dt.

(c) State what value of an initial guess for $y'(x=0)$ led to the final solution.

My initial guess for $y'(x=0)$ was 0.0. The actual, converged value was 1.03455.

(d) Include a copy of the shootinput.m input file you created.

```
function dydt = odeivpn(x,y,nvec);
c=[-1.0,-2.0,2.0,0.0];
dydt(1) = y(2);
dydt(2) = c(1)*y(2) + c(2)*y(1) + c(3)*sin(x) + c(4);
```

Problem (4)

Find an application from your own experience or a classical problem in your own field of research that results in a ODE boundary value problem that can be solved using the shooting method.

(a) Describe the physical problem from which the equation arises. Describe it in sufficient detail that an engineer from a different discipline could understand it.

(b) Write the ODE(s). Write a complete set of reasonable boundary conditions.

(c) If possible, analytically solve for the solution(s). Otherwise numerically solve the for the solution.

(d) Plot the solution.

(e) Explain the physical significance of the solution(s) and its behavior.