ChE/MSE 505
Midterm Examination
Administered: Friday, October 26, 2007

Problem (1)

Consider the system of two linear ODES.

$$\frac{dx_1}{dt} = 8x_1 + 9x_2 - 6$$
$$\frac{dx_2}{dt} = 5x_1 + 6x_2 - 3$$

Determine the location, type and stability of the critical point.

Problem (2)

Consider the system of two nonlinear AES.

$$f_1(x_1, x_2) = 8x_1 + 3x_2^2 - 3$$
$$f_2(x_1, x_2) = 2x_1^2 - 6x_2 - 6$$

Find the approximate value of the root near [0,0].

Solutions:

Problem (1)

Consider the system of two linear ODES.

$$\frac{dx_1}{dt} = 8x_1 + 9x_2 - 6$$
$$\frac{dx_2}{dt} = 5x_1 + 6x_2 - 3$$

Determine the location, type and stability of the critical point.

In order to determine the location of the critical point, we solve

$$\underline{\underline{A}}\underline{x} = \underline{b}$$

$$\begin{bmatrix} 8 & 9 \\ 5 & 6 \end{bmatrix} \underline{x} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

$$\det\left(\underline{\underline{A}}\right) = 8 \cdot 6 - 9 \cdot 5 = 3$$

$$\underline{\underline{A}}^{-1} = \frac{1}{\det\left(\underline{\underline{A}}\right)} \begin{bmatrix} 6 & -9 \\ -5 & 8 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -\frac{5}{3} & \frac{8}{3} \end{bmatrix}$$

$$\underline{x} = \underline{\underline{A}}^{-1}\underline{b} = \begin{bmatrix} 2 & -3 \\ -\frac{5}{3} & \frac{8}{3} \end{bmatrix} \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 12-9 \\ -10+8 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

The critical point is at (3,-2).

In order to determine the type and stability of the critical point, we need the eigenvalues of $\underline{\underline{A}}$.

$$\underline{\underline{A}} - \lambda \underline{\underline{I}} = \begin{bmatrix} 8-\lambda & 9 \\ 5 & 6-\lambda \end{bmatrix}$$

$$(8-\lambda)(6-\lambda) - 45 = 0$$

$$\lambda^2 - 14\lambda + 48 - 45 = \lambda^2 - 14\lambda + 3 = 0$$

$$\lambda = \frac{14 \pm \sqrt{14^2 - 4 \cdot 1 \cdot 3}}{2 \cdot 1} = \frac{14 \pm \sqrt{184}}{2} \approx 13.78 \,\&\, 0.22$$

Eigenvalues are purely real. Therefore the type of critical point is an improper node. The eigenvalues are both positive, therefore the critical point is unstable.

Problem (2)

Consider the system of two nonlinear AES.

$$f_1(x_1, x_2) = 8x_1 + 3x_2^2 - 3$$
$$f_2(x_1, x_2) = 2x_1^2 - 6x_2 - 6$$

Find the approximate value of the root near [0,0].

To determine the location of the root, we must solve,

$$f_1 = 8x_1 + 3x_2^2 - 3 = 0$$

$$f_2 = 2x_1^2 - 6x_2 - 6 = 0$$

Since this is a system of nonlinear equations, we must use an iterative method. We will use the Newton-Raphson method. The Jacobian and residual are:

$$\underline{\underline{J}} = \begin{bmatrix} 8 & 6x_2 \\ 4x_1 & -6 \end{bmatrix}$$

$$\underline{R} = \begin{bmatrix} 8x_1 + 3x_2^2 - 3 \\ 2x_1^2 - 6x_2 - 6 \end{bmatrix}$$

Take as an initial guess $\underline{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

First Iteration.

$$\underline{\underline{J}} = \begin{bmatrix} 8 & 0 \\ 0 & -6 \end{bmatrix} \qquad \det(\underline{\underline{J}}) = -48 \qquad \underline{\underline{J}}^{-1} = \frac{1}{-48}\begin{bmatrix} -6 & 0 \\ 0 & 8 \end{bmatrix}$$

$$\underline{R} = \begin{bmatrix} -3 \\ -6 \end{bmatrix}$$

$$\underline{\delta x} = -\underline{\underline{J}}^{-1}\underline{R} = -\frac{1}{-48}\begin{bmatrix} -6 & 0 \\ 0 & 8 \end{bmatrix}\begin{bmatrix} -3 \\ -6 \end{bmatrix} = -\frac{1}{-48}\begin{bmatrix} 18 \\ -48 \end{bmatrix} = \begin{bmatrix} \frac{3}{8} \\ -1 \end{bmatrix}$$

$$\underline{x}^{new} = \underline{x}^{old} + \underline{\delta x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{3}{8} \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{3}{8} \\ -1 \end{bmatrix}$$

Second Iteration.

$$\underline{\underline{J}} = \begin{bmatrix} 8 & -6 \\ \frac{3}{2} & -6 \end{bmatrix} \qquad \det\left(\underline{\underline{J}}\right) = -48 + 9 = -39 \qquad \underline{\underline{J}}^{-1} = \frac{1}{-39}\begin{bmatrix} -6 & 6 \\ -\frac{3}{2} & 8 \end{bmatrix}$$

$$\underline{R} = \begin{bmatrix} 8\left(\frac{3}{8}\right) + 3(1) - 3 \\ 2\left(\frac{9}{64}\right) - 6(-1) - 6 \end{bmatrix} = \begin{bmatrix} 3 \\ \frac{9}{32} \end{bmatrix}$$

$$\underline{\delta x} = -\underline{\underline{J}}^{-1}\underline{R} = -\frac{1}{-39}\begin{bmatrix} -6 & 6 \\ -\frac{3}{2} & 8 \end{bmatrix}\begin{bmatrix} 3 \\ \frac{9}{32} \end{bmatrix} = \begin{bmatrix} -0.4183 \\ -0.0577 \end{bmatrix}$$

$$\underline{x}^{new} = \underline{x}^{old} + \underline{\delta x} = \begin{bmatrix} \frac{3}{8} \\ -1 \end{bmatrix} + \begin{bmatrix} -0.4183 \\ -0.0577 \end{bmatrix} = \begin{bmatrix} -0.0433 \\ -1.0577 \end{bmatrix}$$

Third Iteration.

$$\underline{\underline{J}} = \begin{bmatrix} 8 & 6(-1.0577) \\ 4(-0.0433) & -6 \end{bmatrix} = \begin{bmatrix} 8 & -6.3462 \\ -0.1731 & -6 \end{bmatrix}$$

$$\det\left(\underline{\underline{J}}\right) = -49.0984 \qquad \underline{\underline{J}}^{-1} = \begin{bmatrix} 0.1222 & -0.1293 \\ -0.0035 & -0.1629 \end{bmatrix}$$

$$\underline{R} = \begin{bmatrix} 8(-0.0433) + 3(-1.0577)^2 - 3 \\ 2(-0.0433)^2 - 6(-1.0577) - 6 \end{bmatrix} = \begin{bmatrix} 0.01 \\ 0.3499 \end{bmatrix}$$

$$\underline{\delta x} = -\underline{\underline{J}}^{-1}\underline{R} = -\begin{bmatrix} 0.1222 & -0.1293 \\ -0.0035 & -0.1629 \end{bmatrix}\begin{bmatrix} 0.01 \\ 0.3499 \end{bmatrix} = \begin{bmatrix} 0.0440 \\ 0.0570 \end{bmatrix}$$

$$\underline{x}^{new} = \underline{x}^{old} + \underline{\delta x} = \begin{bmatrix} -0.4183 \\ -0.0577 \end{bmatrix} + \begin{bmatrix} 0.0440 \\ 0.0570 \end{bmatrix} = \begin{bmatrix} 0.0007 \\ -1.0006 \end{bmatrix} \approx \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

The root is at $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$. (We can see that this exactly satisfies the system of equations.)