Midterm Examination
Administered: Monday, October 24, 2005

Problem (1)

Consider the system of two linear ODES.

$$\frac{dx_1}{dt} = 8x_1 + 9x_2 - 3$$
$$\frac{dx_2}{dt} = 5x_1 + 6x_2 - 6$$

Determine the location, type and stability of the critical point.

Problem (2)

Consider the system of two nonlinear ODES.

$$\frac{dx_1}{dt} = 8x_1 + 3x_2^2 - 3$$
$$\frac{dx_2}{dt} = 2x_1^2 - 6x_2 - 6$$

Determine the approximate location, type and stability of the critical point near [0,0].

Solutions:

Problem (1)

Consider the system of two linear ODES.

$$\frac{dx_1}{dt} = 8x_1 + 9x_2 - 3$$
$$\frac{dx_2}{dt} = 5x_1 + 6x_2 - 6$$

Determine the location, type and stability of the critical point.

In order to determine the location of the critical point, we solve

$$\underline{\underline{A}}\,\underline{x} = \underline{b}$$

$$\begin{bmatrix} 8 & 9 \\ 5 & 6 \end{bmatrix} \underline{x} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$\det\left(\underline{\underline{A}}\right) = 8 \cdot 6 - 9 \cdot 5 = 3$$

$$\underline{\underline{A}}^{-1} = \frac{1}{\det\left(\underline{\underline{A}}\right)} \begin{bmatrix} 6 & -9 \\ -5 & 8 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -\frac{5}{3} & \frac{8}{3} \end{bmatrix}$$

$$\underline{x} = \underline{\underline{A}}^{-1} \underline{b} = \begin{bmatrix} 2 & -3 \\ -\frac{5}{3} & \frac{8}{3} \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 - 18 \\ -5 + 16 \end{bmatrix} = \begin{bmatrix} -12 \\ 11 \end{bmatrix}$$

In order to determine the type and stability of the critical point, we need the eigenvalues of $\underline{\underline{A}}$.

$$\underline{\underline{A}} - \lambda \underline{\underline{I}} = \begin{bmatrix} 8 - \lambda & 9 \\ 5 & 6 - \lambda \end{bmatrix}$$

$$(8 - \lambda)(6 - \lambda) - 45 = 0$$

$$\lambda^2 - 14\lambda + 48 - 45 = \lambda^2 - 14\lambda + 3 = 0$$

$$\lambda = \frac{14 \pm \sqrt{14^2 - 4 \cdot 1 \cdot 3}}{2 \cdot 1} = \frac{14 \pm \sqrt{184}}{2} \approx 13.78 \,\&\, 0.22$$

Eigenvalues are purely real. Therefore the type of critical point is an improper node. The eigenvalues are both positive, therefore the critical point is unstable.

Problem (2)

Consider the system of two nonlinear ODES.

$$\frac{dx_1}{dt} = 8x_1 + 3x_2^2 - 3$$
$$\frac{dx_2}{dt} = 2x_1^2 - 6x_2 - 6$$

Determine the location, type and stability of the critical point near [0,0].

To determine the location of the critical point, we must solve,

$$f_1 = 8x_1 + 3x_2^2 - 3 = 0$$

$$f_2 = 2x_1^2 - 6x_2 - 6 = 0$$

Since this is a system of nonlinear equations, we must use an iterative method. We will use the Newton-Raphson method. The Jacobian and residual are:

$$\underline{\underline{J}} = \begin{bmatrix} 8 & 6x_2 \\ 4x_1 & -6 \end{bmatrix}$$

$$\underline{R} = \begin{bmatrix} 8x_1 + 3x_2^2 - 3 \\ 2x_1^2 - 6x_2 - 6 \end{bmatrix}$$

Take as an initial guess $\underline{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

First Iteration.

$$\underline{\underline{J}} = \begin{bmatrix} 8 & 0 \\ 0 & -6 \end{bmatrix} \qquad \det\left(\underline{\underline{J}}\right) = -48 \qquad \underline{\underline{J}}^{-1} = \frac{1}{-48}\begin{bmatrix} -6 & 0 \\ 0 & 8 \end{bmatrix}$$

$$\underline{R} = \begin{bmatrix} -3 \\ -6 \end{bmatrix}$$

$$\underline{\delta x} = -\underline{\underline{J}}^{-1}\underline{R} = -\frac{1}{-48}\begin{bmatrix} -6 & 0 \\ 0 & 8 \end{bmatrix}\begin{bmatrix} -3 \\ -6 \end{bmatrix} = -\frac{1}{-48}\begin{bmatrix} 18 \\ -48 \end{bmatrix} = \begin{bmatrix} \frac{3}{8} \\ -1 \end{bmatrix}$$

$$\underline{x}^{new} = \underline{x}^{old} + \underline{\delta x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{3}{8} \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{3}{8} \\ -1 \end{bmatrix}$$

Second Iteration.

$$\underline{\underline{J}} = \begin{bmatrix} 8 & -6 \\ \frac{3}{2} & -6 \end{bmatrix} \qquad \det(\underline{\underline{J}}) = -48 + 9 = -39 \qquad \underline{\underline{J}}^{-1} = \frac{1}{-39}\begin{bmatrix} -6 & 6 \\ -\frac{3}{2} & 8 \end{bmatrix}$$

$$\underline{R} = \begin{bmatrix} 8\left(\frac{3}{8}\right) + 3(1) - 3 \\ 2\left(\frac{9}{64}\right) - 6(-1) - 6 \end{bmatrix} = \begin{bmatrix} 3 \\ \frac{9}{32} \end{bmatrix}$$

$$\underline{\delta x} = -\underline{\underline{J}}^{-1}\underline{R} = -\frac{1}{-39}\begin{bmatrix} -6 & 6 \\ -\frac{3}{2} & 8 \end{bmatrix}\begin{bmatrix} 3 \\ \frac{9}{32} \end{bmatrix} = \begin{bmatrix} -0.4183 \\ -0.0577 \end{bmatrix}$$

$$\underline{x}^{new} = \underline{x}^{old} + \underline{\delta x} = \begin{bmatrix} \frac{3}{8} \\ -1 \end{bmatrix} + \begin{bmatrix} -0.4183 \\ -0.0577 \end{bmatrix} = \begin{bmatrix} -0.0433 \\ -1.0577 \end{bmatrix}$$

Third Iteration.

$$\underline{\underline{J}} = \begin{bmatrix} 8 & 6(-1.0577) \\ 4(-0.0433) & -6 \end{bmatrix} = \begin{bmatrix} 8 & -6.3462 \\ -0.1731 & -6 \end{bmatrix}$$

$$\det(\underline{\underline{J}}) = -49.0984 \qquad \underline{\underline{J}}^{-1} = \begin{bmatrix} 0.1222 & -0.1293 \\ -0.0035 & -0.1629 \end{bmatrix}$$

$$\underline{R} = \begin{bmatrix} 8(-0.0433) + 3(-1.0577)^2 - 3 \\ 2(-0.0433)^2 - 6(-1.0577) - 6 \end{bmatrix} = \begin{bmatrix} 0.01 \\ 0.3499 \end{bmatrix}$$

$$\underline{\delta x} = -\underline{\underline{J}}^{-1}\underline{R} = -\begin{bmatrix} 0.1222 & -0.1293 \\ -0.0035 & -0.1629 \end{bmatrix}\begin{bmatrix} 0.01 \\ 0.3499 \end{bmatrix} = \begin{bmatrix} 0.0440 \\ 0.0570 \end{bmatrix}$$

$$\underline{x}^{new} = \underline{x}^{old} + \underline{\delta x} = \begin{bmatrix} -0.4183 \\ -0.0577 \end{bmatrix} + \begin{bmatrix} 0.0440 \\ 0.0570 \end{bmatrix} = \begin{bmatrix} 0.0007 \\ -1.0006 \end{bmatrix} \approx \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

The critical point is at $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$. (We can see that this exactly satisfies the system of equations.)

In order to determine the type and stability of the critical point, we need the eigenvalues of $\underline{\underline{J}}$.

$$\underline{\underline{J}} = \begin{bmatrix} 8 & 6(-1) \\ 4(0) & -6 \end{bmatrix} = \begin{bmatrix} 8 & -6 \\ 0 & -6 \end{bmatrix}$$

$$\underline{\underline{J}} - \lambda \underline{\underline{I}} = \begin{bmatrix} 8-\lambda & -6 \\ 0 & -6-\lambda \end{bmatrix}$$

$$(8-\lambda)(-6-\lambda) - 0 = 0$$

$$\lambda = 8 \,\&\, -6$$

Eigenvalues are purely real. One eigenvalue is positive and the other eigenvalue is negative, therefore the critical point is a saddle point. All saddle points are unstable.