Analytical Study of Stability of Systems of ODEs

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date begun: September 1999
updated: October 10, 2005

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I. 2 linear ODEs with constant coefficients

The material in Section I of these lecture notes corresponds to the contents of Chapter 3 Sections, 3.3 and 3.4, pages 162-175, Kreyszig, 8th Edition.

This material applies to systems of linear ODEs with constant coefficients. This is a very limited subset of problems but it is worth seeing what rigorous criteria for stability we can achieve. We will proceed by examining a system of 2 linear ODEs. The work will apply for a system of n linear ODEs but, it is much easier to visualize in two dimensions.

We have a system of ODEs of the form

$$\frac{d\underline{y}}{dx} = \underline{\underline{A}}\,\underline{y} + \underline{b} \tag{I.1}$$

and we have an initial condition of the form:

$$\underline{y}(x = x_o) = \underline{y}_o \tag{I.2}$$

This system of equations has a critical point at $\underline{y}_c = \underline{y}(x_c)$, where $\underline{y}_c$ satisfies the condition:

$$\frac{dy_2}{dy_1} = \frac{dy_2/dx}{dy_1/dx} = \frac{a_{21}y_1 + a_{22}y_2 + b_2}{a_{11}y_1 + a_{12}y_2 + b_1} = \frac{0}{0} \tag{I.3}$$

which yields

$$\underline{y}_c = -\underline{\underline{A}}^{-1}\underline{b} = \begin{bmatrix} -\frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}}\left(\frac{a_{21}b_1 - a_{11}b_2}{\det(A)}\right) \\ \frac{a_{21}b_1 - a_{11}b_2}{\det(A)} \end{bmatrix} \tag{I.4}$$

or, equivalently

$$\underline{y}_c = -\underline{\underline{A}}^{-1}\underline{b} = \begin{bmatrix} -\frac{b_2}{a_{21}} - \frac{a_{22}}{a_{21}}\left(\frac{a_{21}b_1 - a_{11}b_2}{\det(A)}\right) \\ \frac{a_{21}b_1 - a_{11}b_2}{\det(A)} \end{bmatrix} \tag{I.5}$$

Clearly the critical points are the values of $\underline{y}$ when the derivatives of $\underline{y}$ are zero. However, when the derivatives are zero, the function is constant, so these are the steady-state, or equilibrium, or long-time (depending on the problem) solutions to the ODE.

All trajectories pass through the critical point, including the eigenvectors.

Five Types of Critical Points

There are five types of critical points.

- Improper nodes
- Proper nodes
- Saddle points
- Centers
- Spiral points

Kreyszig gives the following way to determine which type of critical point a system has. For a 2x2 matrix, write the characteristic equation:

$$(a_{11}-\lambda)(a_{22}-\lambda)-a_{21}a_{12}=0 \quad \text{(I.6)}$$

with eigenvalues given by

$$(a_{11}-\lambda)(a_{22}-\lambda)-a_{21}a_{12}=0 \quad \text{(I.6)}$$

$$\lambda^2-(a_{11}+a_{22})\lambda+\det(A)=0 \quad \text{(I.7)}$$

$$\lambda=\frac{(a_{11}+a_{22})\pm\sqrt{(a_{11}+a_{22})^2-4\det(A)}}{2} \quad \text{(I.8)}$$

Kreyszig chooses to write this as

$$\lambda=\frac{p\pm\sqrt{\Delta}}{2} \quad \text{(I.9)}$$

where $p=a_{11}+a_{22}$, $q=\det(A)$, $\Delta=p^2-4q$

Case 1. If $q>0$ and $\Delta\geq 0$

We have either a proper or improper node.
We have eigenvalues that are purely real.

Case 2. If $q<0$ (forcing $\Delta>0$)

We have a saddle point.
We have eigenvalues that are purely real.

Case 3. If $p=0$ and $q>0$ (forcing $\Delta<0$)

We have a center.
We have eigenvalues that are purely imaginary.

Case 4. If $p\neq 0$ and $\Delta<0$

We have a spiral point.
We have eigenvalues that are complex (both nonzero real and imaginary parts).

Kreyszig has examples of each of the five types of critical points on pages 164-166.

Stability

In addition to the type of critical point, we might like to know the stability of the critical point.

Case 1. If $p < 0$ and $q > 0$

The critical point is stable and attractive.

We have eigenvalues with negative real parts. The imaginary part can be zero (giving a purely real eigenvalue) or positive and negative (giving complex conjugates for the eigenvalues).

The critical point is either a node or a spiral point.

Case 2. If $p = 0$ and $q > 0$

The critical point is stable.

We have eigenvalues with zero real parts. The eigenvalues are purely imaginary complex conjugates.

The critical point is a center.

Case 3. If $p > 0$ or $q < 0$

The critical point is unstable.

We have at least one eigenvalue with positive real part.

The critical point is a spiral point, node, or saddle point.

See Figure 88, on page 173 of Kreyszig.

What's the point?

The point of studying critical point type and stability is to determine a priori what sort of qualitative behavior you can expect from the solution to a system of ODEs.

Starting at any initial condition, you can expect the solution to move through x toward a stable attractor but away from an unstable point. For example, you can expect the solution to move in a spiral toward the critical point if the eigenvalues have negative real parts and nonzero imaginary parts.

This concept will be illustrated in the homework assignments.

II. N linear ODEs with constant coefficients

When we unambiguously outlined stability analysis for a system of 2 linear ODEs with constant coefficients. What if we have n linear ODEs?

We have some immediate difficulties. What is the definition of a critical point? For a system of 2 ODEs, it is

$$\frac{dy_2}{dy_1} = \frac{dy_2/dx}{dy_1/dx} = \frac{0}{0} \tag{II.1}$$

By analogy, the definition of a critical point for a system of n linear ODEs is

$$\frac{dy_i}{dx} = 0 \text{ for all } 1 \le i \le n \tag{II.2}$$

Let's take the 3x3 example.

$$\underline{y}_c = -\underline{\underline{A}}^{-1}\underline{b} \tag{II.3}$$

So that we see that a system of linear equations, regardless of the size, has only one critical point.

In order to determine the type and stability of the critical point, we can no longer use Kreyszig's, p, q, and Δ criteria because the characteristic equation is no longer quadratic. However, we can still use the eigenvalue criteria.

If the eigenvalues all have negative real parts, the critical point is a stable attractor.

If any one of the eigenvalues has a positive real part, the critical point is unstable.

If all of the eigenvalues have zero real parts, the critical point is a center.

If any of the eigenvalues have nonzero imaginary parts, there will be a spiraling nature to the critical point.

Example:

Consider the system of linear ODEs:

$$\frac{d\underline{y}}{dx} = \underline{\underline{A}}\underline{y} + \underline{b} \tag{I.1}$$

where

$$\underline{\underline{A}} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \text{ and } \underline{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The eigenvalues and eigenvectors of A are:

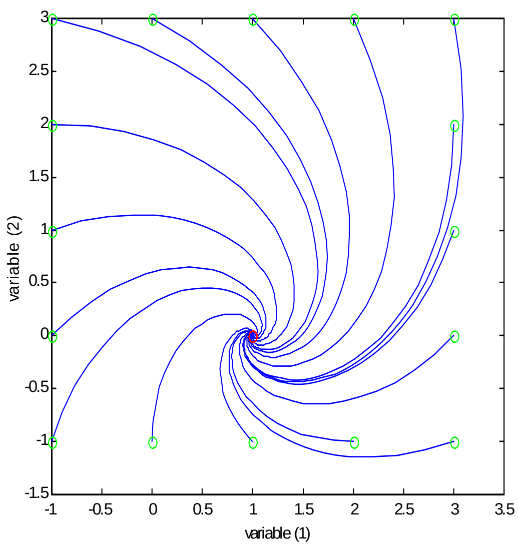
$$\underline{\underline{\Lambda}} = \begin{bmatrix} -1+i & 0 & 0 \\ 0 & -1-i & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad \underline{\underline{W}}_c = \begin{bmatrix} 1 & 1 & 0 \\ i & -i & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

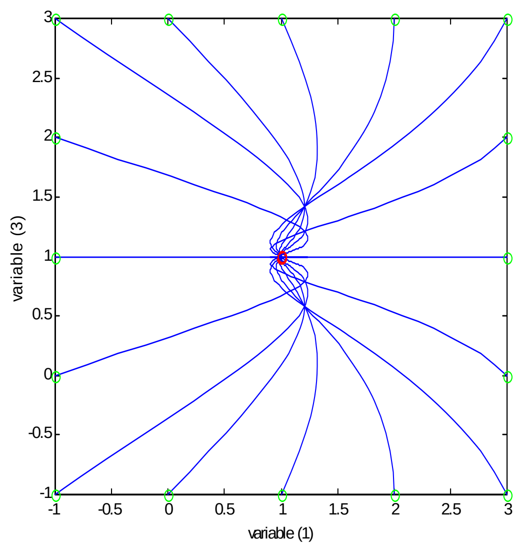
All of the real parts of the eigenvalues are negative. Therefore, we should expect an attractor. Some of the eigenvalues have nonzero imaginary parts, therefore we should expect a spiral.

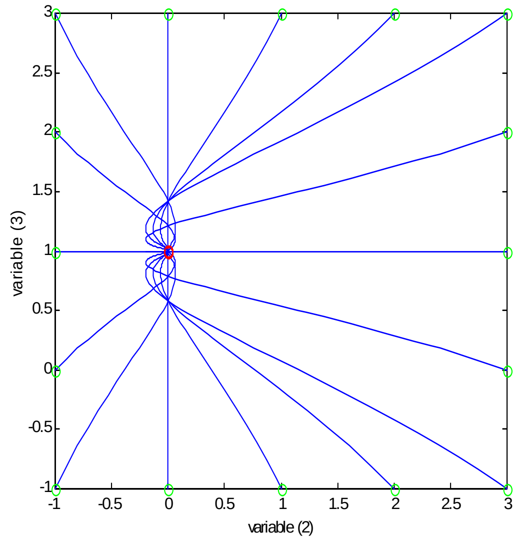
We see that the critical point is given as:

$$\underline{y}_c = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Some trajectories in various planes of 3-D space are given below.







One can see that we have spiraling, attractive behavior in three-dimensions, as we expected from the qualitative features of the eigenvalues.

III. Linear ODEs with variable coefficients

We have a system of ODEs of the form

$$\frac{d\underline{y}}{dx} = \underline{\underline{A}}(x)\underline{y} + \underline{b}(x) \tag{III.1}$$

and we have an initial condition of the form:

$$\underline{y}(x = x_o) = \underline{y}_o \tag{III.2}$$

Again, we will have a critical point at $\underline{y}_c = \underline{y}(x_c)$, where $\underline{y}_c$ satisfies the condition:

$$\frac{dy_2}{dy_1} = \frac{dy_2/dx}{dy_1/dx} = \frac{a_{21}(x)y_1 + a_{22}(x)y_2 + b_2(x)}{a_{11}(x)y_1 + a_{12}(x)y_2 + b_1(x)} = \frac{0}{0} \tag{III.3}$$

which yields

$$\underline{y}_c(x) = \underline{\underline{A}}^{-1}\underline{b} = \begin{bmatrix} -\frac{b_1(x)}{a_{11}(x)} - \frac{a_{12}(x)}{a_{11}(x)}\left(\frac{a_{21}(x)b_1(x) - a_{11}(x)b_2(x)}{\det(A(x))}\right) \\ \frac{a_{21}(x)b_1(x) - a_{11}(x)b_2(x)}{\det(A(x))} \end{bmatrix} \tag{III.4}$$

So it becomes clear that for every value of the dependent variable x, there is a critical point that is a function of x.

Likewise, the eigenvalues and eigenfunctions are now functions of x.

At any instant in x, the criteria for types and stability of critical points holds. If the type and stability hold for a range of x, then we can expect the associated behavior of the solution to hold over that range.

Example:

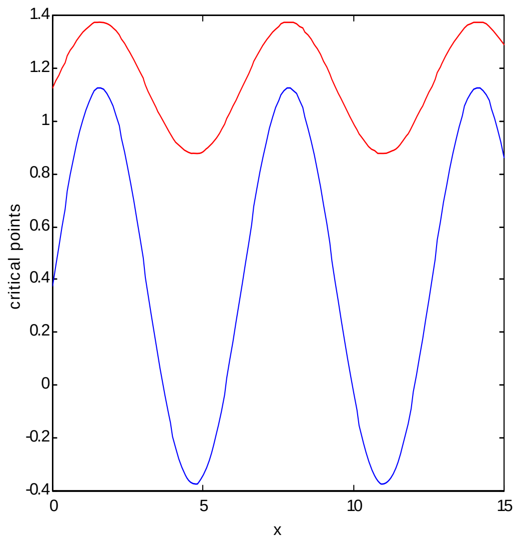
$$\underline{\underline{A}} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}$$

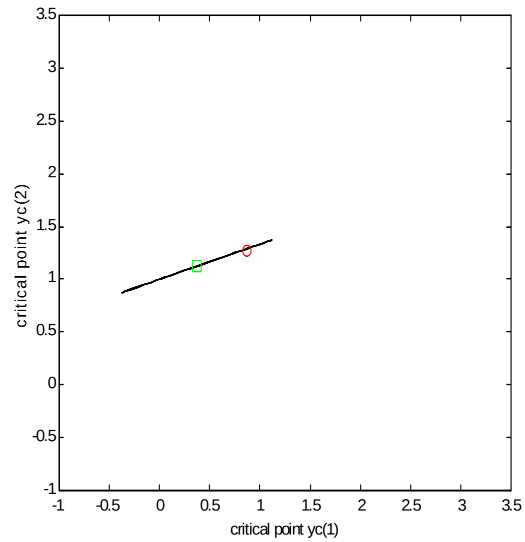
$$\underline{b} = \begin{bmatrix} 2\sin(x) \\ 3 \end{bmatrix}$$

The eigenvalues of A are $\underline{\underline{\Lambda}} = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix}$ and the eigenvectors $\underline{\underline{W}}_c = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

The eigenvalues are purely real, and negative. Therefore, we expect a stable attractor, with node-like behavior.

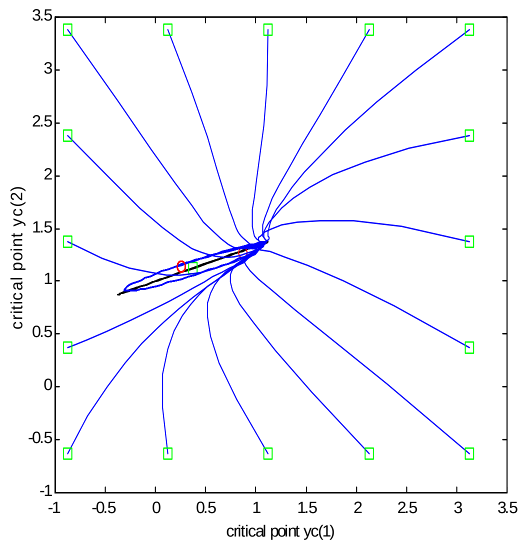
Using equation (III.4) we find that the critical points as a function of x look like:





In the figure on the left, we have plotted the y(1) component of the critical point as a function of x in red and the y(2) component of the critical point as a function of x in blue. In the figure on the right, we have plotted y(2) vs y(1) as parametric functions of x. Let's call this curve of critical points the critical path.

Below we show several solutions to the ODE, starting from different initial conditions. The starting points of each line are indicated by green squares. The ending points are indicated by red circles.



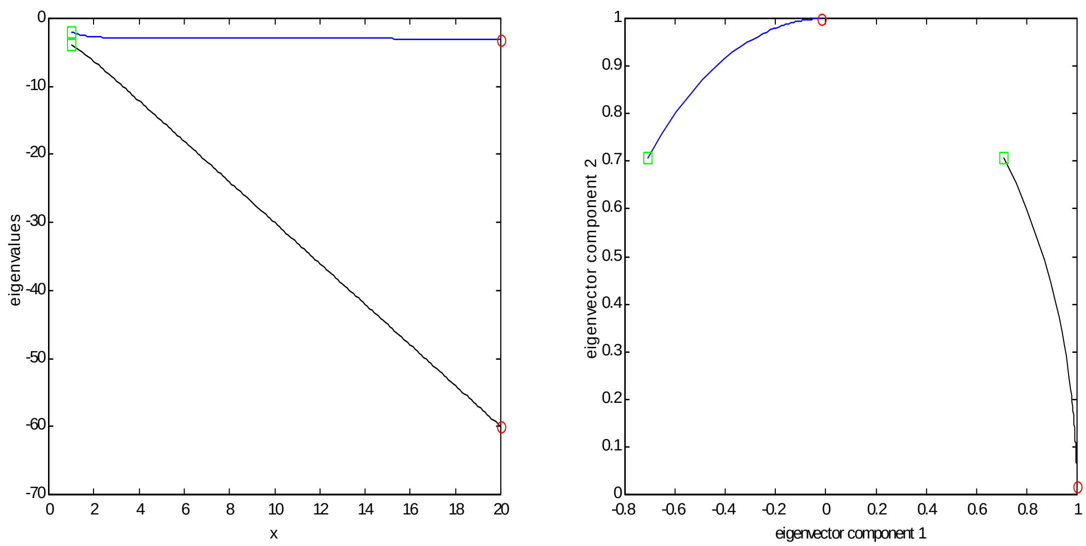
We see that the solution is indeed an attractor. All points lead to the critical path. Moreover, they lead to the critical path in a node-like way, without spiraling. The curious behavior is at the center where we do find a cyclical steady state. This cyclical steady state is due to the sine function in b(x). It is interesting that the trajectories never actually fall on the black line indicated by the critical path, but rather form a cycle about it. This must be due to the fact that the ODEs at time x are heading toward a solution defined by b(x). However, at some incremental time later, x', the solution has now moved and is defined by b(x'). Thus, the path of the ODE must be altered. The solution can be said to lag behind the critical path. All solutions find the same lag, as indicated by the fact that regardless of the initial condition, the final position (in this case plotted at x=15) is the same.

Example:

$$\underline{\underline{A}} = \begin{bmatrix} -3x & 1 \\ 1 & -3 \end{bmatrix}$$

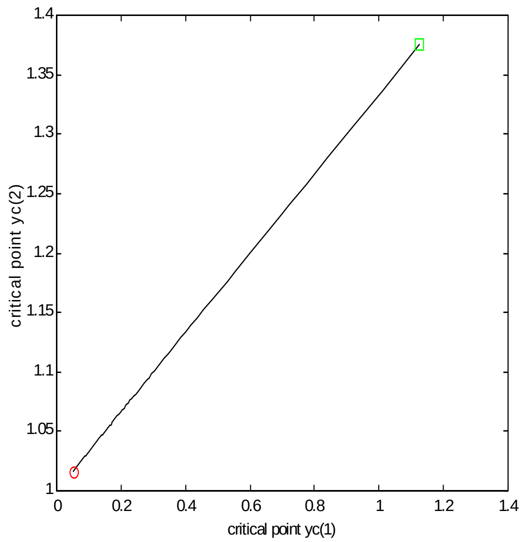
$$\underline{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

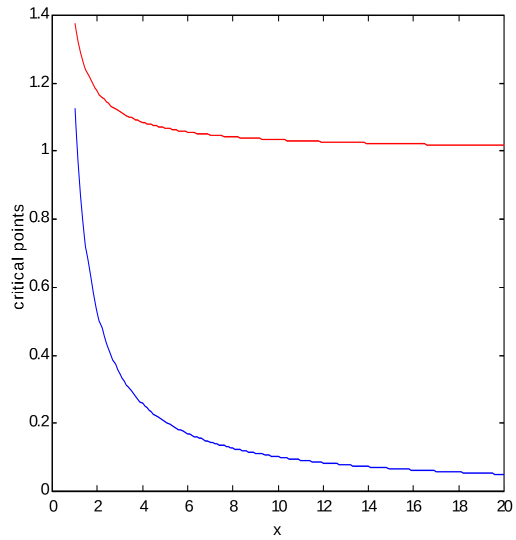
The eigenvalues and eigenvectors are now functions of x, as shown below:



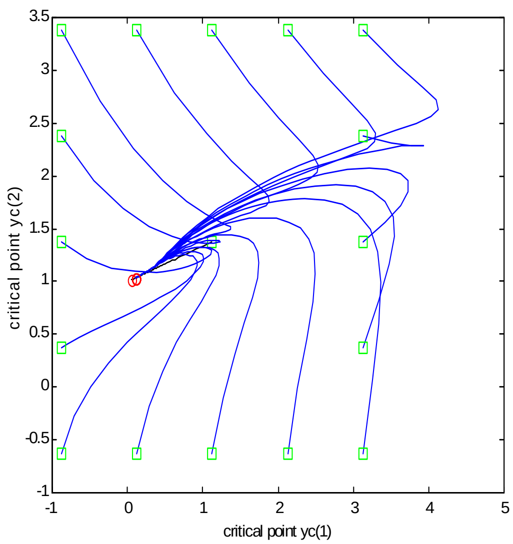
We can see that the eigenvalues start small and negative; one linearly decreases and the other appears to exponentially decrease. The eigenvectors appear to be approaching asymptotic values of [1,0] and [0,1] respectively. The point is that the eigenvalues are always negative and always purely real.

Plots of the critical path are given below, both as a function of x and parametrically.





Lastly, we show trajectories for various initial conditions.



This figure shows that we have a stable node for a critical point. We would expect this because our eigenvalues are real and negative. The only difference between this case and the case where the matrix A is constant is that now our critical point is mobile. The trajectories follow along behind it.

IV. Non-Linear ODEs

Most of the world's problems are non-linear. What good does our previous analysis of linear ODEs do for us? Well, it tells us the five types of critical points. It tells us that the critical points are the steady-state solutions. It tells us that we can understand the behavior of a system of ODEs by looking at a phase portrait.
With the generalized definition of the critical points,

$$\frac{dy_i}{dx} = 0 \text{ for all } 1 \le i \le n \tag{II.2}$$

we can still determine critical points in the nonlinear case.

Example:

$$\frac{dy_1}{dx} = y_1(x)^2 - 3y_2(x) - 4$$
$$\frac{dy_2}{dx} = -3y_1(x) + y_2(x)$$

A critical point of this system of equations is $\underline{y}_c = \begin{bmatrix} -0.4244 \\ -1.2733 \end{bmatrix}$

In order to determine the eigenvalues, we need a matrix. In this case, since the ODEs are nonlinear, we cannot directly write the ODEs in matrix form. Instead, we will take a Taylor Series expansion of the ODEs about the critical point.

$$\frac{dy_1}{dx} = y_1(x)^2 - 3y_2(x) - 4 = f_1(y_1(x), y_2(x))$$
$$\frac{dy_2}{dx} = -3y_1(x) + y_2(x) = f_2(y_1(x), y_2(x))$$

We expand the two functions in a Taylor series about the critical point and truncate after the linear terms,

$$f_1(y_1(x), y_2(x)) = f_1(y_{1c}, y_{2c}) + \left.\frac{df_1}{dy_1}\right|_{\underline{y}_c}(y_1(x) - y_{1c}) + \left.\frac{df_1}{dy_2}\right|_{\underline{y}_c}(y_2(x) - y_{2c})$$
$$f_2(y_1(x), y_2(x)) = f_2(y_{1c}, y_{2c}) + \left.\frac{df_2}{dy_1}\right|_{\underline{y}_c}(y_1(x) - y_{1c}) + \left.\frac{df_2}{dy_2}\right|_{\underline{y}_c}(y_2(x) - y_{2c})$$

We have linearized the functions and the linearized ODEs are now,

$$\frac{dy_1}{dx} = f_1(y_{1c}, y_{2c}) + \left.\frac{df_1}{dy_1}\right|_{\underline{y}_c} (y_1(x) - y_{1c}) + \left.\frac{df_1}{dy_2}\right|_{\underline{y}_c} (y_2(x) - y_{2c})$$
$$\frac{dy_2}{dx} = f_2(y_{1c}, y_{2c}) + \left.\frac{df_2}{dy_1}\right|_{\underline{y}_c} (y_1(x) - y_{1c}) + \left.\frac{df_2}{dy_2}\right|_{\underline{y}_c} (y_2(x) - y_{2c})$$

We can write this in matrix notation as

$$\frac{d\underline{y}}{dx} = \underline{\underline{J}}\,\underline{y} + \underline{b}$$

where $\underline{\underline{J}}$ is the Jacobian, which has the definition,

$$\underline{\underline{J}} \equiv \begin{bmatrix} \left.\frac{df_1}{dy_1}\right|_{\underline{y}_c} & \left.\frac{df_1}{dy_2}\right|_{\underline{y}_c} \\ \left.\frac{df_2}{dy_1}\right|_{\underline{y}_c} & \left.\frac{df_2}{dy_2}\right|_{\underline{y}_c} \end{bmatrix}$$

and $\underline{b}$ is a vector of constants,

$$\underline{b} = \begin{bmatrix} f_1(y_{1c}, y_{2c}) - \left.\frac{df_1}{dy_1}\right|_{\underline{y}_c} y_{1c} - \left.\frac{df_1}{dy_2}\right|_{\underline{y}_c} y_{2c} \\ f_2(y_{1c}, y_{2c}) - \left.\frac{df_2}{dy_1}\right|_{\underline{y}_c} y_{1c} - \left.\frac{df_2}{dy_2}\right|_{\underline{y}_c} y_{2c} \end{bmatrix}$$

The Jacobian used here is the exact same Jacobian that is used in the Newton-Raphson method. Once we have linearized the ODE, we can use the straightforward procedure to determine the eigenvalues and eigenvectors of the Jacobian.

$$\det\left(\underline{\underline{J}} - \lambda \underline{\underline{I}}\right) = 0$$

We now proceed as before in the eigenanalysis. For this particular, example problem, we have

$$\underline{\underline{J}} = \begin{bmatrix} 2y_{1c} & -3 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -0.8488 & -3 \\ -3 & 1 \end{bmatrix}$$

The eigenvalues of the Jacobian matrix are

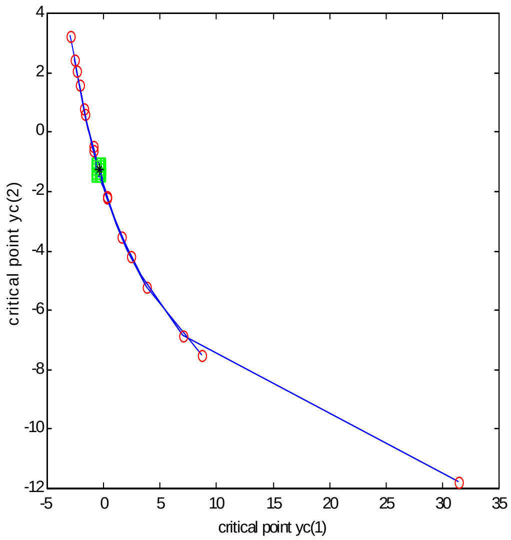
$$\underline{\underline{\Lambda}} = \begin{bmatrix} -3.0636 & 0 \\ 0 & 3.2148 \end{bmatrix}$$

The eigenvectors of the Jacobian matrix are

$$\underline{\underline{W}} = \begin{bmatrix} -0.8045 & -0.5939 \\ -0.5939 & 0.8045 \end{bmatrix}$$

The eigenvalues are real, therefore we either have a node or a saddle point. The eigenvalues are not all of the same sign. Therefore, we have a saddlepoint. All saddlepoints are unstable. We should expect that this problem has a critical point that behaves like a saddle point. In truth, this behavior should be observed very close to the critical point. It may extend beyond that.

Plots of trajectories starting from initial conditions near the critical point yield:



Clearly, this is an unstable critical point. The starting points (green squares) all lead away from the critical point (black star) to their respective ending points at x=1.0 (red circles). The eigenvectors leading away from the critical point are not straight lines since the problem is nonlinear.

Example:

$$\frac{dy_1}{dx} = -y_1(x)^2 + y_2(x) + 1$$
$$\frac{dy_2}{dx} = -y_1(x) - y_2(x)$$

A critical point of this system of equations is $\underline{y}_c = \begin{bmatrix} 0.6180 \\ -0.6180 \end{bmatrix}$

In order to determine the eigenvalues, we again linearize the system of ODEs with a Taylor series expansion. The Jacobian of the linearized problem is

$$\underline{\underline{J}} \equiv \begin{bmatrix} \left.\frac{df_1}{dy_1}\right|_{\underline{y}_c} & \left.\frac{df_1}{dy_2}\right|_{\underline{y}_c} \\ \left.\frac{df_2}{dy_1}\right|_{\underline{y}_c} & \left.\frac{df_2}{dy_2}\right|_{\underline{y}_c} \end{bmatrix} = \begin{bmatrix} -2y_{1c} & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -1.2360 & 1 \\ -1 & -1 \end{bmatrix}$$

The eigenvalues of the Jacobian matrix are

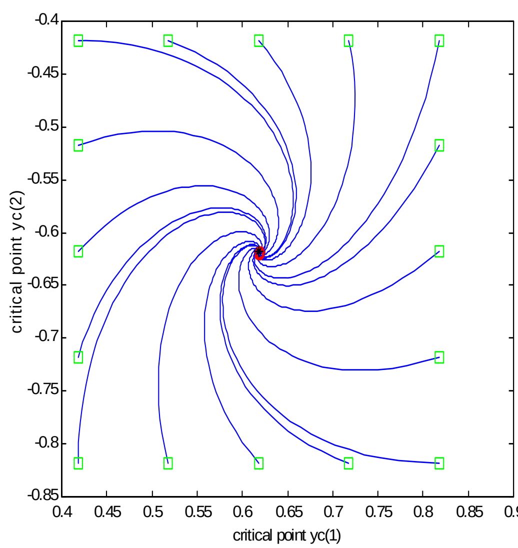
$$\underline{\underline{\Lambda}} = \begin{bmatrix} -1.1180 + 0.9930i & 0 \\ 0 & -1.1180 - 0.9930i \end{bmatrix}$$

The eigenvectors of the Jacobian matrix are

$$\underline{\underline{W}} = \begin{bmatrix} 0.7071 & 0.7071 \\ 0.0834 + 0.7022i & 0.0834 - 0.7022i \end{bmatrix}$$

The eigenvalues are complex, therefore we have a spiral point. The real component of the eigenvalues are less than zero. Therefore, the point is stable. We should observe the behavior of a stable spiral point, at least near the critical point.

Plots of trajectories starting from initial conditions near the critical point yield:



Clearly, this is a stable attractor with a spiraling nature.

Non-linear equations can have more than one critical point.

Example:

```
x = y(1);                    % extent of reaction
T = y(2);                    % Temperature K
Cin = 3.0;                   % inlet concentration mol/l
C = Cin*(1-x);               % concentration
Q = 60e-3;                   % volumetric flowrate l/s
R = 8.314;                   % gas constant J/mol/K
Ea = 62800;                  % activation energy J/mol
ko = 4.48e+6;                % reaction rate prefactor 1/s
k = ko*exp(-Ea/(R*T));       % reaction rate constant 1/s
V = 18;                      % reactor volume l
Cp = 4.19e3;                 % heat capacity J/kg/K
 Tin = 298;             % inlet feed temperature K
 Tref = 298;            % thermodynamic reference temperature K
 DHr = -2.09e5;         % heat of rxn J/mol
 rho = 1.0;             % density kg/l
 dydt(1) = 1/V*(Q*Cin - Q*C - k*C*V);  % mass balance mol/s
 dydt(2) = 1/(Cp*rho*V)*(Q*Cp*rho*Tin  - Q*Cp*rho*T -
          DHr*k*C*V);         % NRG balance J/s
 dydt(1) = -1/Cin*dydt(1);  % convert from concentration to
                            extent
```

These are the design equations for a continuously stirred-tank reactor with a single first-order exothermic reaction, operating under adiabatic conditions.

The unknowns are the extent of reaction and the the temperature.

This problem has three steady states. The critical points of this system of equations are

$$\underline{y}_c = \begin{bmatrix} x \\ T \end{bmatrix},\ \underline{y}_{c,1} = \begin{bmatrix} 0.0159 \\ 300.4 \end{bmatrix},\ \underline{y}_{c,2} = \begin{bmatrix} 0.3335 \\ 347.9 \end{bmatrix},\text{ and } \underline{y}_{c,3} = \begin{bmatrix} 0.9828 \\ 445.1 \end{bmatrix}$$

In order to determine the eigenvalues, we again linearize the system of ODEs with a Taylor series expansion. In this case, since we have more than one critical point, we must evaluate the Jacobian at each of the steady states. The Jacobian of the linearized problem is

$$\underline{\underline{J}} \equiv \begin{bmatrix} \left.\frac{df_1}{dy_1}\right|_{\underline{y}_c} & \left.\frac{df_1}{dy_2}\right|_{\underline{y}_c} \\ \left.\frac{df_2}{dy_1}\right|_{\underline{y}_c} & \left.\frac{df_2}{dy_2}\right|_{\underline{y}_c} \end{bmatrix} = \begin{bmatrix} \left.\frac{df_1}{dx}\right|_{\underline{y}_c} & \left.\frac{df_1}{dT}\right|_{\underline{y}_c} \\ \left.\frac{df_2}{dx}\right|_{\underline{y}_c} & \left.\frac{df_2}{dT}\right|_{\underline{y}_c} \end{bmatrix} = \begin{bmatrix} \left.\frac{df_1}{dC}\right|_{\underline{y}_c} \frac{dC}{dx} & \left.\frac{df_1}{dT}\right|_{\underline{y}_c} \\ \left.\frac{df_2}{dC}\right|_{\underline{y}_c} \frac{dC}{dx} & \left.\frac{df_2}{dT}\right|_{\underline{y}_c} \end{bmatrix}$$

$$\underline{\underline{J}} = \begin{bmatrix} -\frac{Q}{V} - k & \frac{kE_a}{C_{in}RT^2} \\ \frac{\Delta H_r C_{in} k}{C_p \rho} & -\frac{Q}{V} - \frac{\Delta H_r C k E_a}{C_p \rho R T^2} \end{bmatrix}$$

We evaluate this Jacobian at each critical point and get the eigenvalues. For the first critical point, we have

$$\underline{\underline{J}} = \begin{bmatrix} -0.0034 & 0.0000 \\ -0.0081 & -0.0027 \end{bmatrix}$$

$$\underline{\underline{\Lambda}} = \begin{bmatrix} -0.0034 & 0 \\ 0 & -0.0027 \end{bmatrix}$$

$$\underline{\underline{W}} = \begin{bmatrix} -0.0866 & -0.0021 \\ -0.9962 & -1.0000 \end{bmatrix}$$

The eigenvalues of the first critical point are real and negative. Therefore, the first critical point will behave like a stable improper node.

For the second critical point, we have

$$\underline{\underline{J}} = \begin{bmatrix} -0.0050 & 0.0000 \\ -0.2495 & 0.0070 \end{bmatrix}$$

$$\underline{\underline{\Lambda}} = \begin{bmatrix} -0.0042 & 0 \\ 0 & 0.0063 \end{bmatrix}$$

$$\underline{\underline{W}} = \begin{bmatrix} -0.0452 & -0.0031 \\ -0.9990 & -1.0000 \end{bmatrix}$$

The eigenvalues of the second critical point are real. One is positive and one is negative. Therefore, the second critical point will behave like a saddle point, which is always unstable.

For the third and final critical point, we have

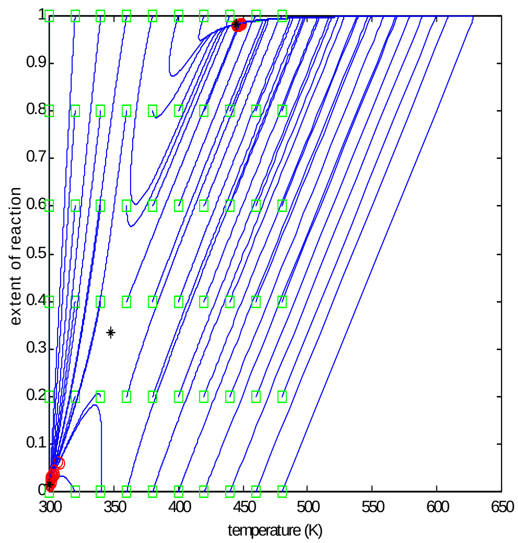
$$\underline{\underline{J}} = \begin{bmatrix} -0.1944 & 0.0024 \\ -28.5878 & 0.0154 \end{bmatrix}$$

$$\underline{\underline{\Lambda}} = \begin{bmatrix} -0.0895 + 0.2417i & 0 \\ 0 & -0.0895 - 0.2417i \end{bmatrix}$$

$$\underline{\underline{W}} = \begin{bmatrix} 0.0037 - 0.0085i & 0.0037 + 0.0085i \\ 1.0000 & 1.0000 \end{bmatrix}$$

The eigenvalues of the third critical point are complex. The real components of the eigenvalues are negative. Therefore, the third critical point will behave like a stable spiral point.

Some trajectories, based on different initial conditions (different initial concentrations and reactor temperatures) are shown below. The trajectories start at green squares and end at red circles. The time that transpired along each trajectory is 1 minute.



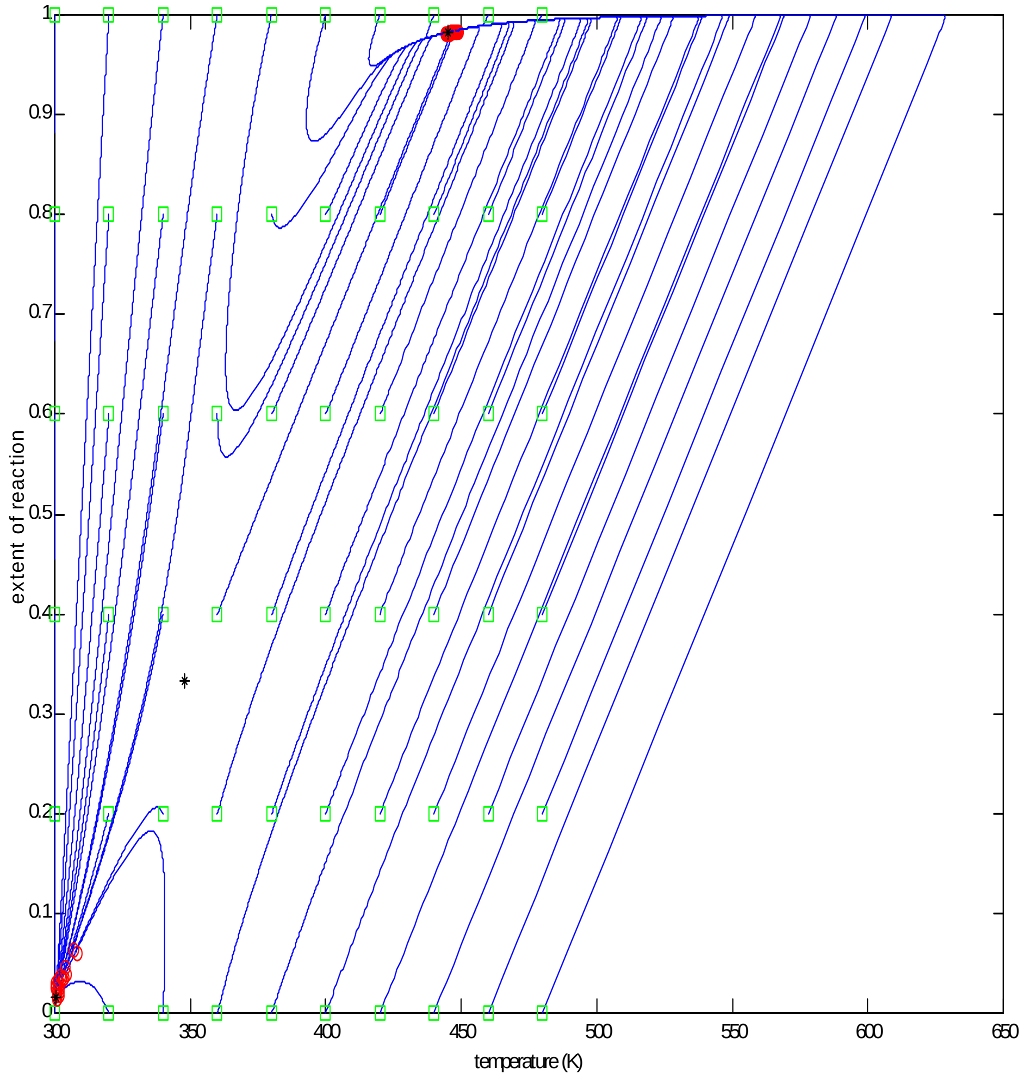
(a) initial temperatures = 300 < T < 500
initial extent of reactions 0 < x < 1
duration of trajectory = 1200 sec
(larger version of plot available on last page of this section)

From the trajectory plots given above we can determine the nature of the critical points (steady state solutions in this example). The low-conversion/low-temperature and the high-conversion/high-temperature solutions are stable attractors. The intermediate solution is an unstable node. The eigenvalues associated with the attractors are less than zero. At least one eigenvalue associated with the unstable node is negative.

We can also see some qualitative information about the system. We can define roughly the basins of attraction for the two attractors. For the coarse grid we used, any initial temperature of 380 K or higher converged to the high critical point. Any initial temperature of 340 K or lower converged to the low critical point. For initial temperatures of 360K, those with high initial extents of reaction proceeded to the low root; those with low initial extents of reaction converged to the high root.

The trajectories that led to the low root, approached with a final tangents that appeared to be nearly parallel to the difference vector between the low and middle root. The trajectories that led to the high root, approached with two different final tangents. The first seemed to be nearly parallel to the difference vector between the high and middle root. The second, which most of the trajectories followed, appeared to be roughly perpendicular to the the first.

Some initial conditions with low initial extent of reaction and low temperature, proceeded through temperatures higher than the high root on their way to the high root. This is because the reactor is full of unreacted product. It reacts initially, which, since the reaction is exothermic, heats up the reactor. It then takes some time for new feed to enter and cool the reactor to its steady state temperature.


extent of reaction
temperature (K)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
300
350
400
450
500
550
600
650