Exam IV: Administered: May 7, 2001
70 points (7 problems @ 10 points each)

Problem (1).
We need to buy pumps to move 60 gallons of water per second in a process line for use for one year. We can buy 3 pumps with a 20 gal/sec capacity or 2 with a 30 gal/sec capacity. The probability that a 20 gal/sec pump fails within a year is 9% and the probability that a 30 gal/sec pump fails within a year is 3%. Catastrophic failure (reactor explosion) occurs if all pumps fail. Assume the pumps are independent.

(a)What is the probability that all three 20-gps pumps fail within a year?
(b)What is the probability that both 30-gps pumps fail within a year?
(c) Is using three 20-gps or two 30-gps pumps better to prevent catastrophic failure?

Solution:

Since the pumps are independent, the intersection of all pumps failing is the product of the individual failure probabilities.

$$P(A \cap B) = P(A)P(B) \qquad \text{for A and B independent.}$$

$$P(A \cap B \cap C) = P(A)P(B)P(C) \qquad \text{for A, B, and C independent.}$$

(a)What is the probability that all three 20-gps pumps fail within a year?

$$P(\text{three 20 - gps fail}) = P(\text{one 20 - gps fails})^3 = 0.09^3 = 0.000729$$

(b)What is the probability that both 30-gps pumps fail within a year?

$$P(\text{two 30 - gps fail}) = P(\text{one 30 - gps fails})^2 = 0.03^2 = 0.0009$$

(c) Is using three 20-gps or two 30-gps pumps better to prevent catastrophic failure?

$$P(\text{three 20 - gps fail}) < P(\text{two 30 - gps fail})$$

So, using three 20-gps pumps is safer, despite the fact that individually they have a higher probability of failing.

Problem (2)
Perform one complete Newton-Raphson iteration on the system of equations:

$$y = \ln(x) \qquad\qquad 3y^2 + \sqrt{x} = 10$$

Use (x,y) = (2,2) as your initial guess.
Along the way, present the Jacobian, Residual, determinant, inverse, and new estimate of [x,y].

solution:

$$f_1(x,y) = \ln(x) - y = 0 \qquad\qquad f_2(x,y) = 10 - 3y^2 - \sqrt{x} = 0$$

$$\underline{\underline{J}} = \begin{bmatrix} \frac{1}{x} & -1 \\ -\frac{1}{2\sqrt{x}} & -6y \end{bmatrix} \quad \underline{R} = \begin{bmatrix} \ln(x) - y \\ 10 - 3y^2 - \sqrt{x} \end{bmatrix}$$

$$\underline{\underline{J}}(x = 2, y = 2) = \begin{bmatrix} \frac{1}{2} & -1 \\ -\frac{1}{2\sqrt{2}} & -12 \end{bmatrix},$$

$$\underline{R}(x = 2, y = 2) = \begin{bmatrix} \ln(x) - y \\ 10 - 3y^2 - \sqrt{x} \end{bmatrix} = \begin{bmatrix} \ln(2) - 2 \\ -2 - \sqrt{2} \end{bmatrix} \approx \begin{bmatrix} -1.30685 \\ -3.41421 \end{bmatrix}$$

$$\det(J) = j_{11}j_{22} - j_{21}j_{12} = (\frac{1}{2})(-12) - (-\frac{1}{2\sqrt{2}})(-1) = -6 - \frac{1}{2\sqrt{2}} \approx -6.3536$$

$$\underline{\underline{J}}^{-1} = \frac{1}{\det(J)}\begin{bmatrix} j_{22} & -j_{12} \\ -j_{21} & j_{11} \end{bmatrix} = \frac{1}{-6 - \frac{1}{2\sqrt{2}}}\begin{bmatrix} \frac{1}{2} & -1 \\ -\frac{1}{2\sqrt{2}} & -12 \end{bmatrix}$$

$$\underline{\delta x} = -\underline{\underline{J}}^{-1}\underline{R} = -\frac{1}{-6 - \frac{1}{2\sqrt{2}}}\begin{bmatrix} \frac{1}{2} & -1 \\ -\frac{1}{2\sqrt{2}} & -12 \end{bmatrix}\begin{bmatrix} \ln(2) - 2 \\ -2 - \sqrt{2} \end{bmatrix} \approx \begin{bmatrix} 1.9309 \\ -0.3414 \end{bmatrix}$$

$$\underline{x}^{(1)} = \underline{x}^{(0)} + \underline{\delta x} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 1.9309 \\ -0.3414 \end{bmatrix} = \begin{bmatrix} 3.9309 \\ 1.6586 \end{bmatrix}$$

Problem (3)
We are developing a process where the quality of the feedstock is important. Poor quality feedstock can result in unacceptable product. A vendor for the feedstock provides us with 18 samples. He *claims* that the population mean purity of the feed stock is 0.80 and *claims* that the population standard deviation is 0.003. We run the 18 samples through our own lab and find a sample mean purity of 0.803 with a sample standard deviation of 0.004. Based on this information, answer the following questions.
(a) What PDF is appropriate for determining a confidence interval on the variance?
(b) Find the lower limit on a 96% confidence interval on the variance.
(c) Find the upper limit on a 96% confidence interval on the variance.
(d) Is the vendor's claim legitimate?
(e) If our maximum allowable standard deviation is 0.0045, can we be 96% confident that the vendor's feedstock is adequate?

Solution:

(a) What PDF is appropriate for determining a confidence interval on the variance?
Chi-squared distribution for the confidence interval on the variance.

(b) Find the lower limit on a 96% confidence interval on the variance.
(c) Find the upper limit on a 96% confidence interval on the variance.

$$\alpha = \frac{1 - C.I.}{2} = 0.02 \qquad v = n - 1 = 17$$

$$\chi^2_{\alpha} = 30.995 \text{ from table A.5} \qquad \chi^2_{1-\alpha} = 7.255 \text{ from table A.5}$$

$$s^2 = 0.004^2 = 1.6 \cdot 10^{-5}$$

$$P\left[\frac{(n-1)s^2}{\chi^2_{\alpha}} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{1-\alpha}}\right] = 1 - 2\alpha$$

$$P\left[8.776 \cdot 10^{-6} < \sigma^2 < 37.491 \cdot 10^{-6}\right] = 0.96$$

(d) Is the vendor's claim legitimate?

The vendor claimed that $\sigma^2 = 0.003^2 = 9.0 \cdot 10^{-6}$. Since this value falls within our confidence interval, his claim is legitimate.

(e) If our maximum allowable standard deviation is 0.0045, can we be 96% confident that the vendor's feedstock is adequate?

Our maximum allowable deviation is $s^2 = 0.0045^2 = 2.025 \cdot 10^{-5}$. This value lies within the confidence interval. Since this value is our maximum acceptable value, there are points within this confidence interval that we cannot accept. Therefore, we cannot be confident that the vendor's feedstock is adequate.

Problem (4)

On your first day at work as the new process engineer for a plant producing a liquid fungicide product, you are shown a closet packed with ten years of process analysis in the form of moldy strip charts and daily quality control measurements of the concentration of active (fungus-killing) ingredient in the liquid product. The plant manager tells you, the young engineer fresh from school, to "analyze" the data and answer two questions for her.

Her questions are:

(1) What fraction of the product has a concentration of active ingredient less than 0.05 mol/liter?

(2) What is the concentration of active ingredient for which 90% of the product is greater than?

What would you do in this situation? (Quitting is not an option; you have to pay for your new car.) For this exam, answer these questions:

(a) What information would need to extract from the data?

(b) How would you get the information in (a) from the plant's data?

(c) What PDF, would you use to answer questions (1) and (2)?

(d) Outline with equations or references to tables, how you would obtain answers to questions (1) and (2), assuming you had the necessary information in (a)?

Solution:

(c) Basically we need something that gives us the probability of a successful/defective process. We could use the binomial distribution, but with ten years of data, our number of trials, N, is huge. As N becomes large, the binomial distribution is approximated by the normal distribution. Therefore, I would use the normal distribution.

(a) When using the normal distribution, one requires the population mean and the population variance.

(b) In order to obtain the population mean and variance, I would take the data and have the manager buy some software that could convert the analog stripcharts into a table. I would assign the engineering intern at the plant all the menial labor required in this conversion of the data from analog to digital form.

Once the data was in digital form, we would have a bunch of concentration data points. We would calculate the sample mean and the sample variance, using equations of the form:

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i \qquad \text{and} \qquad s^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left(x_i - \bar{x}\right)^2$$

We then assume that the population mean and variance are well approximated by the sample mean and variance. (After all, we have ten years of data.) So, $\mu \approx \bar{x}$ and $\sigma \approx s$.

(d) The manager's questions are of the standard type "find P given z" (question (1)) and "find z given P" (question (2)).

(1) What fraction of the product has a concentration of active ingredient less than 0.05 mol/liter?

$$z = \frac{x-\mu}{\sigma} = \frac{0.05-\mu}{\sigma}$$

$$p(x < 0.05) = p(z < \frac{0.05-\mu}{\sigma})$$

For a specific value of the population mean and variance, we can go to the standard normal distribution table in the back of WMM and find the probability p.

(2) What is the concentration of active ingredient for which 90% of the product is greater than?

$$p(z > z_{lo}) = 0.90$$

$$p(z < z_{lo}) = 1 - 0.90 = 0.10$$

From table A.3 in WMM, we find that $z_{lo} = -1.28$

$$z = \frac{x-\mu}{\sigma} = -1.28$$

For a particular value of the population mean and variance, we can solve this equation for x. This x is the concentration which 90% of the product is greater than.

Problem (5)

In solving the solution to $\underline{\underline{A}}\underline{x} = \underline{b}$, where $\underline{\underline{A}} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 4 & 3 & 4 \end{bmatrix}$ and $\underline{b} = \begin{bmatrix} 3 \\ 5 \\ 11 \end{bmatrix}$, we find the following information on the determinant, rank, and reduced row echelon form of the A|b augmented matrix.

$$\det(\underline{\underline{A}}) = 0 \qquad \text{rank}(\underline{\underline{A}}) = 2 \quad \text{rank}(\underline{\underline{A}} \,|\, \underline{b}) = 2 \qquad \text{rref}\left(\underline{\underline{A}} \,|\, \underline{b}\right) = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(a) Does the inverse of A exist?
(b) How many solutions exist to $\underline{\underline{A}}\underline{x} = \underline{b}$?
(c) If infinite solutions exist, find the solution $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}$.

Solution:

(a) Does the inverse of A exist?

No. The determinant of A is zero.

(b) How many solutions exist to $\underline{\underline{A}}\underline{x} = \underline{b}$?

Since the det(A) = 0, & rank(A|b) = rank(A), there are an infinite number of solutions.

(c) If infinite solutions exist, find the solution $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}$.

We take the rref(A|b) and convert it to a new 2x2 matrix equation of the form $\underline{\underline{A}}\underline{x} = \underline{b}$:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 - x_3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

det(A) = 1, $\underline{\underline{A}}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underline{\underline{A}}^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ so the solution is $\underline{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Problem (6)
Consider a reactive separation process using a membrane of thickness W, operating at steady state. On one side of the membrane, you have a mixture with a concentration of ethanol of C(x=0) = 10.0 mol/liter. On the other side of the membrane the concentration of ethanol is C(x=W) = 5.0 mol/liter. Inside the membrane, ethanol diffuses with diffusion coefficient, D, and ethanol is consumed via a chemical reaction with rate constant, k. The differential equation which describes the *steady state* concentration profile in the membrane can be derived from a mass balance and is given as

$$0 = -D\left(\frac{d^2C}{dx^2}\right) - kC$$

Your task is to find the *steady state* concentration profile within the membrane.

(a) Identify the independent variable
(b) Identify the dependent variable
(c) Identify the O.D.E. as linear or nonlinear
(d) Identify the order of the differential equation
(e) Identify the type of problem: Initial-Value Problem or Boundary-Value Problem
(f) If necessary, transform a single n^{th}-order equation into a system of n first-order equations.
(g) Name and describe the standard numerical algorithm needed to solve this problem
(h) Predict the difficulty/ease of obtaining a solution with the method from (g)

Solution:

(a) Identify the independent variable: x
(b) Identify the dependent variable: C
(c) Identify the O.D.E. as linear or nonlinear: linear
(d) Identify the order of the differential equation: second
(e) Identify the type of problem: Initial-Value Problem or Boundary-Value Problem: BVP
(f) If necessary, transform a single n^{th}-order equation into a system of n first-order equations.

$$y_1 = C \text{ and } y_2 = \frac{dC}{dx} \text{ so } \quad \frac{dy_1}{dx} = y_2 \text{ and } \frac{dy_2}{dx} = -\frac{k}{D}y_1$$

(g) Name and describe the standard numerical algorithm needed to solve this problem: The shooting method.

(h) Predict the difficulty/ease of obtaining a solution with the method from (g)
Since the problem is linear the shooting method will converge in exactly 3 iterations, if you use linear interpolation as your method for getting a new estimate of $y_2(x = 0)$.

Problem (7)

In Computer Project 2, you plotted the transient behavior of the reactor temperature of the non-isothermal, non-adiabatic reactor as a function of time. Qualitatively reproduce that plot here with a sketch. Describe the physical phenomena responsible for increases, decreases, maxima, minima, and/or plateaus in your plot.

Solution:

The initial gradual increase in temperature is due to the reactant feed streams coming in at hotter temperatures then the initial temperature of the solvent in the reactor.

The second, steeper increase in temperature is due to the exothermic reaction taking off.

The peak in temperature, if observed, occurs because the reactor starts at low temperature, where the reactants build up. As the temperature increases due to exothermic reaction, the concentration of the reactants decrease. Once the reactants have decreased enough, the reaction rate drops, and the temperature peaks and begins to drop.

The plateau in temperature indicates that the process has approached steady-state operation.