Exam IV: Administered: May 9, 2000
120 points

Problem (1) (20 minutes - 20 points)
Perform one complete Newton-Raphson iteration on the system of equations:

$$y = \exp(x)$$

$$y^2 + x^3 = 10$$

Use (x,y) = (1,1) as your initial guess.

solution:

$$f_1(x,y) = \exp(x) - y = 0$$

$$f_2(x,y) = 10 - y^2 - x^3 = 0$$

$$\underline{\underline{J}} = \begin{bmatrix} \exp(x) & -1 \\ -3x^2 & -2y \end{bmatrix} \quad \underline{R} = \begin{bmatrix} \exp(x) - y \\ 10 - y^2 - x^3 \end{bmatrix}$$

$$\underline{\underline{J}}(x = 1, y = 1) = \begin{bmatrix} e & -1 \\ -3 & -2 \end{bmatrix}, \quad \underline{R}(x = 1, y = 1) = \begin{bmatrix} \exp(x) - y \\ 10 - y^2 - x^3 \end{bmatrix} = \begin{bmatrix} e - 1 \\ 8 \end{bmatrix}$$

$$\det(J) = j_{11}j_{22} - j_{21}j_{12} = (e)(-2) - (-3)(-1) = -2e - 3$$

$$\underline{\underline{J}}^{-1} = \frac{1}{\det(J)} \begin{bmatrix} j_{22} & -j_{12} \\ -j_{21} & j_{11} \end{bmatrix} = \frac{1}{-2e - 3} \begin{bmatrix} -2 & 1 \\ 3 & e \end{bmatrix}$$

$$\underline{\delta x} = -\underline{\underline{J}}^{-1}\underline{R} = -\frac{1}{-2e - 3} \begin{bmatrix} -2 & 1 \\ 3 & e \end{bmatrix} \begin{bmatrix} e - 1 \\ 8 \end{bmatrix} = -\begin{bmatrix} -2(e - 1) + 8 \\ 3(e - 1) + 8e \end{bmatrix} \approx \begin{bmatrix} 0.5409 \\ 3.1886 \end{bmatrix}$$

$$\underline{x}^{(1)} = \underline{x}^{(0)} + \underline{\delta x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0.5409 \\ 3.1886 \end{bmatrix} = \begin{bmatrix} 1.5409 \\ 4.1886 \end{bmatrix}$$

Problem (2) (20 points)
Consider the data that describes the concentration of product (mole/liter) when comparing two different company's feed-stocks. Each experiment was done with 9 replicates.
We input the data (18 data points) into the MATLAB program *anova_1factor.m* and obtained the following output:

Ho: all treatments are equal
Reject Ho if 0.46 >> f(1, 16)
Hypothesis NOT Rejected for 90 percent confidence interval (0.46 < 4.50)

pvalue = 5.21e-001

90 percent C.I. on the 1 treatment: 1.20e+000 < 1.28e+000 < 1.36e+000
90 percent C.I. on the 2 treatment: 1.24e+000 < 1.32e+000 < 1.40e+000

90 percent C.I. on the 1 - 2 treatment diff.: -1.59e-001 < -4.44e-002 < 7.03e-002

Based on this output, answer the following questions.

(a) Do the companies offer significantly different feed-stocks?

(b) At what confidence interval does the null hypothesis switch from being rejected to not rejected?

(c) If vendor 1 claims that his feed-stock will yield a product concentration 0.07 mol/liter higher than vendor 2, is this claim valid?

(d) If vendor 2 claims that his feed-stock will yield a product concentration 0.1 mol/liter higher than vendor 1, is this claim valid?

(e) Explain your answers to (c) and (d).

Solution:

(a) Do the companies offer significantly different feed-stocks?

No. The null hypothesis was not rejected. Treatments are the same.

(b) At what confidence interval does the null hypothesis switch from being rejected to not rejected?

$$\text{C.I.} = 1 - 2p = 1 - 2 \cdot 0.521 = -0.042 = -4.2\%$$

This negative value means that there is no C.I. where the null hypothesis will switch.

(c) If vendor 1 claims that his feed-stock will yield a product concentration 0.07 mol/liter higher than vendor 2, is this claim valid?

Yes. 0.07 is within the confidence interval of the difference of means.

(d) If vendor 2 claims that his feed-stock will yield a product concentration 0.1 mol/liter higher than vendor 1, is this claim valid?

Yes. -0.1 is within the confidence interval of the difference of means.

(e) Explain your answers to (c) and (d).

Both claims are valid because the claimed increase in product concentration is so small that it falls within the confidence interval for the two (similar) feedstocks.

Problem (3) (20 minutes - 20 points)
From historical data, we know that a process produces a batch of polymer with an average molecular weight of 500,000 and a standard deviation of 10,000.

(a) What is the probability of finding a polymer with a molecular weight less than 480,000?

(b) 75% of the polymers described above have a molecular weight greater than y. Find y.

Solution:

Since only the mean and standard deviation are given, we have no choice but to use the normal distribution.

(a)

$$z = \frac{x - \mu}{\sigma} = \frac{480{,}000 - 500{,}000}{10{,}000} = -2$$

$P(x < 480{,}000) = P(z < -2) = 0.0228$ from Appendix 3 of WMM.

(b)

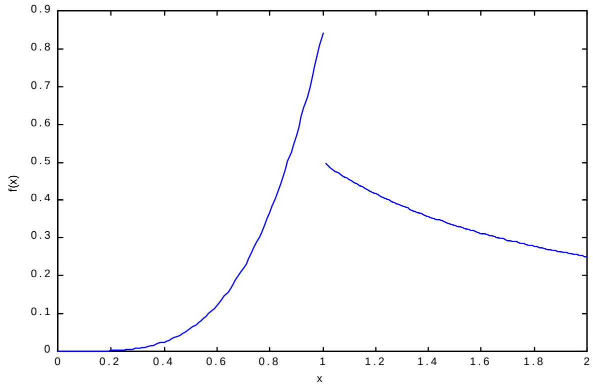
$$P(x > y) = P(z > z_y) = 1 - P(z < z_y) = 0.75$$
$$P(z < z_y) = 0.25$$

$z_y = -0.675$ from Appendix 3 of WMM.

$$z_y = -0.675 = \frac{y - 500{,}000}{10{,}000}$$
$$y = 493{,}250$$

Problem (4) (20 minutes - 20 points)
Consider the function described by the formula and plot below:

$$f(x) = \begin{cases} x^3 \sin(x) & \text{for } x \le 1 \\ \dfrac{1}{2x} & \text{for } x > 1 \end{cases}$$



Integrate this function from 0.5 to 1.5 using the Trapezoidal rule and 2 intervals.

solution:
Since this function is discontinuous. We can only integrate over continuous intervals.

$$\int_{0.5}^{1.5} f(x)dx = \int_{0.5}^{1.0} f(x)dx + \int_{1.0}^{1.5} f(x)dx = \int_{0.5}^{1.0} x^3 \sin(x)dx + \int_{1.0}^{1.5} \frac{1}{2x}dx$$

Approximate each integral as a trapezoid.

$$\int_{0.5}^{1.0} x^3 \sin(x)dx \approx \frac{1}{2}(1.0 - 0.5)\left[0.5^3 \sin(0.5) + 1.0^3 \sin(1.0)\right] = 0.2253$$

$$\int_{1.0}^{1.5} \frac{1}{2x}dx \approx \frac{1}{2}(1.5 - 1.0)\left[\frac{1}{2\cdot(1.0)} + \frac{1}{2\cdot(1.5)}\right] = 0.2083$$

$$\int_{0.5}^{1.5} f(x)dx \approx 0.4336$$

Problem (5) (20 minutes - 20 points)

Consider an nxn matrix, $\underline{\underline{J}}$, with rank = n. Indicate which of any of the following statements are true.

(a) The inverse of $\underline{\underline{J}}$ exists.

(b) At least 2 rows of $\underline{\underline{J}}$ are linearly dependent.

(c) The determinant of $\underline{\underline{J}}$ is non-zero.

(d) There is a unique solution to $\underline{\underline{J}}\underline{x} = \underline{b}$ for any real nx1 vector, $\underline{b}$.

(e) The reduced row echelon form of $\underline{\underline{J}}$ will not have any rows completely filled with zeroes.

(f) The rank of $\underline{\underline{J}}$ is n.

(g) The matrix $\underline{\underline{J}}$ has less than n non-zero eigenvalues.

Solution:

(a) True
(b) False
(c) True
(d) True
(e) True
(f) True
(g) False

Problem (6) (20 minutes - 20 points)

Consider a one-dimensional rod of length L. The end of the rod at x=0 is maintained at a constant temperature T_0. The end of the rod at x=L is maintained at a constant temperature T_L. The rod is metal and has a thermal conductivity, k, density, ρ, and heat capacity, C_p. Between the two ends, the rod loses heat to the surroundings which are at a constant temperature T_S. The ordinary differential equation which describes the steady state temperature profile in the rod can be derived from an energy balance and is given as

$$0 = \frac{k}{\rho C_p}\left(\frac{d^2T}{dx^2}\right) + \frac{hA\left(T_S - T\right)}{\rho C_p V}$$

where A is the surface area of the rod exposed to the surroundings, V is the volume of the rod, and h is the heat transfer coefficient between the rod and surroundings.

Your task is to find the steady state temperature profile.

(a) Identify the independent variable
(b) Identify the dependent variable
(c) Identify the O.D.E. as linear or nonlinear
(d) Identify the order of the differential equation
(e) Identify the type of problem: Initial-Value Problem or Boundary-Value Problem
(f) If necessary, transform a single n^{th}-order equation into a system of n first-order equations.
(g) Name and describe the standard numerical algorithm needed to solve this problem
(h) Predict the difficulty/ease of obtaining a solution with the method from (g)

Solution:

(a) Identify the independent variable: x
(b) Identify the dependent variable: T
(c) Identify the O.D.E. as linear or nonlinear: linear
(d) Identify the order of the differential equation: second
(e) Identify the type of problem: Initial-Value Problem or Boundary-Value Problem: BVP
(f) If necessary, transform a single n^{th}-order equation into a system of n first-order equations.

$$y_1 = T \text{ and } y_2 = \frac{dT}{dx} \text{ so } \quad \frac{dy_1}{dx} = y_2 \text{ and } \frac{dy_2}{dx} = -\frac{hA\left(T_S - y_1\right)}{kV}$$

(g) Name and describe the standard numerical algorithm needed to solve this problem: The shooting method.
(h) Predict the difficulty/ease of obtaining a solution with the method from (g)
Since the problem is linear the shooting method will converge in exactly 3 iterations, if you use linear interpolation as your method for getting a new estimate of $y_2(x=0)$.