Exam I
Administered: Monday, February 7, 2000
24 points

For each problem part: 0 points if not attempted or no work shown,
1 point for partial credit, if work is shown,
2 points for correct numerical value of solution

Problem 1. (10 points)
The plant manager in a fungicide manufacturing plant has been doing an internal study of quality control as a function of operator shift: He has found that, when the A-shift is at work from 6:00 AM to 2:00 PM, there is 99% of the product is Good. When the B-shift is at work from 2:00 PM to 10:00 PM, there is 97% of the product is Good. When the C-shift is at work from 10:00 PM to 6 A.M, there is 93% of the product is Good.

(a) Draw a Venn Diagram of the sample space for a random variable denoting the quality of the product.
(b) What is the probability that the product is Good?
(c) What is the probability that the product is Defective?
(d) What is the probability that the product is Defective and C-shift was responsible?
(e) Given that the product is Defective, what is the probability that the C-shift was responsible?

Solution:
(a) Draw a Venn Diagram of the sample space for a random variable denoting the quality of the product.
A=A-shift, B=B-shift, C=C-shift, G=Good, D=Defective

$A \cap G$	$B \cap G$	$C \cap G$
$A \cap D$	$B \cap D$	$C \cap D$

(b) What is the probability that the product is Good?

The probability of each shift being on is equal, one third of the time. $P(A) = P(B) = P(C) = \frac{1}{3}$

The probability that a product is Good is then a simplified version of the union rule (simplified since there are no interesections of A, B, and C).

$$P(G) = P(A \cap G) + P(B \cap G) + P(C \cap G)$$
$$P(G) = P(G \mid A)P(A) + P(G \mid B)P(B) + P(G \mid C)P(C)$$
$$P(G) = 0.99 \cdot \frac{1}{3} + 0.97 \cdot \frac{1}{3} + 0.93 \cdot \frac{1}{3} = 0.9633$$

(c) What is the probability that the product is Defective?

$$P(G) + P(D) = 1$$
$$P(D) = 1 - P(G) = 1 - 0.9633 = 0.0367$$

(d) What is the probability that the product is Defective and C-shift was responsible?

$$P(C \cap D) + P(C \cap G) = P(C)$$
$$P(C \cap D) = P(C) - P(C \cap G)$$
$$P(C \cap D) = P(C) - P(G \mid C)P(C) = \frac{1}{3} - 0.93 \cdot \frac{1}{3} = \frac{0.07}{3} = 0.233$$

(e) Given that the product is Defective, what is the probability that the C-shift was responsible?

$$P(C\,|\,D) = \frac{P(C \cap D)}{P(D)} = \frac{\frac{0.07}{3}}{0.0367} = 0.6358$$

Problem 2. (4 points)
Consider two independent random variables x and y. The random variable y has mean $\mu_y = 8.00$ and variance $\sigma_y^2 = 4.00$. The random variable x obeys the PDF

$$f(x) = \begin{cases} \left(\frac{e}{4e-4}\right) e^{-\frac{x}{4}} & \text{for } 0 < x < 4 \\ 0 & \text{otherwise} \end{cases}$$

(a) What is the mean of $g(x) = e^{-\frac{3x}{4}}$
(b) What is the mean of $h(x, y) = g(x) + 5y$

Solution:

(a) What is the mean of $g(x) = e^{-\frac{3x}{4}}$

$$\mu_{g(x)} = \int_{-\infty}^{\infty} g(x) f(x,y)\, dx = \int_0^4 e^{-\frac{3x}{4}} \left(\frac{e}{4e-4}\right) e^{-\frac{x}{4}}\, dx = \left(\frac{e}{4e-4}\right) \int_0^4 e^{-x}\, dx$$

$$\mu_{g(x)} = -\left(\frac{e}{4e-4}\right) \left[e^{-x}\right]_0^4 = -\left(\frac{e}{4e-4}\right) \left[e^{-4} - 1\right] = 0.38825$$

(b) What is the mean of $h(x, y) = g(x) + 5y$

$$\mu_{h(x,y)} = \mu_{g(x)} + 5\mu_y = 0.38825 + 5 \cdot 8.00 = 40.38825$$

Problem 3. (10 points)
Consider the joint PDF of two random variables, x and y,

$$f(x, y) = \begin{cases} \frac{4x}{3y} & \text{for } 0 < x < 1,\ y = 1,2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Is the random variable x continuous or discrete?
(b) Is the random variable y continuous or discrete?
(c) Show that this PDF is a legitimate PDF.
(d) Find the probability, $P(0 < x < 0.5, y = 1)$

(e) Find the probability , $P(y = 2)$

Solution:
(a) Is the random variable x continuous or discrete?
X is a continuous random variable because it is defined over a continuous range.

(b) Is the random variable y continuous or discrete?
Y is a discrete random variable because it is defined over a discrete range.

(c) Show that this PDF is a legitimate PDF.

(i) the PDF can never be negative because $\frac{4x}{3y}$ is always nonnegative for positive values of x and y, which are where the PDF is defined.

(ii) the PDF intgrates to 1

$$\sum_{y=1}^{2}\left[\int_{-\infty}^{\infty} f(x,y)\,dx\right] = \sum_{y=1}^{2}\left[\int_{0}^{1} \frac{4x}{3y}\,dx\right] = \sum_{y=1}^{2}\left[\frac{4x^2}{6y}\right]_0^1 = \sum_{y=1}^{2}\frac{4}{6y} = \frac{4}{6\cdot 1} + \frac{4}{6\cdot 2} = 1$$

Alternatively but equivalently, we could write:

$$\int_{-\infty}^{\infty}\sum_{y=1}^{2} f(x,y)\,dx = \int_{0}^{1}\sum_{y=1}^{2}\frac{4x}{3y}\,dx = \int_{0}^{1}\left(\frac{4x}{3\cdot 1} + \frac{4x}{3\cdot 2}\right)dx = \int_{0}^{1} 2x\,dx = \left.\frac{2x^2}{2}\right|_0^1 = 1$$

(d) Find the probability, $P(0 < x < 0.5, y = 1)$

$$P(0 < x < 0.5, y = 1) = \int_{0}^{0.5} f(x, y = 1)\,dx = \int_{0}^{0.5}\frac{4x}{3\cdot 1}\,dx = \left.\frac{4x^2}{6}\right|_0^{0.5} = \frac{1}{6}$$

(e) Find the probability , $P(y = 2)$

If only y is specified, then we don’t care about the value of x so we include all possible values of x.

$$P(0 < x < 1, y = 2) = \int_{0}^{1} f(x, y = 2)\,dx = \int_{0}^{1}\frac{4x}{3\cdot 2}\,dx = \left.\frac{4x^2}{12}\right|_0^{1} = \frac{1}{3}$$