Final Examination
Administered: Thursday, December 16, 1999, 12:30-2:30

Problem (1) (20 minutes - 20 points)
Consider manufacturing plant with two identical compressors, one online and the other a back-up. The mean continuous operating time of a compressor is 1680 hours before maintenance. If the primary compressor breaks down and it takes two weeks to fix it, what is the probability that the back-up compressor will fail before the primary compressor has been repaired?

solution:
The probability that the back-up compressor with a mean life of 1680 hours fails before 2 weeks (336 hours) is given by the exponential distribution:

$$P(t < t_i) = \int_0^{t_i} f_e(t;\beta)dt = \int_0^{t_i} \frac{1}{\beta} e^{-t/\beta}dt = e^{-t/\beta}\Big|_0^{t_i} = 1 - e^{-t_i/\beta} = 0.18127$$

Problem (2) (20 minutes - 20 points)
In the upcoming census, statisticians predict that 2.0% of the U.S. population (of approximately 300 million people) will be omitted from the count. Of those omitted, it is projected that 70% will be minors. The total U.S. population consists of 20% minors. Answer the following questions.

(a) Given that a person is a minor what is the probability that they are omitted from the census?
(b) What is the probability that a person is an adult and is counted in the census?
(c) What is the probability that a person is an adult given that they were counted in the census.

solution:
The solution space is divided into four groups as shown below:

$\text{Minor} \cap \text{Counted}$	$\text{Adult} \cap \text{Counted}$
$\text{Minor} \cap \text{Omitted}$	$\text{Adult} \cap \text{Omitted}$

(a) Given that a person is a minor what is the probability that they are omitted from the census?

$$P(M|O) = 0.7 = \frac{P(O \cap M)}{P(O)} = \frac{P(O \cap M)}{0.02}$$
$$P(O \cap M) = 0.7 \cdot 0.02 = 0.014$$
$$P(O|M) = \frac{P(O \cap M)}{P(M)} = \frac{0.014}{0.2} = 0.07$$

(b) What is the probability that a person is an adult and is included in the census?

All four of the following are legitimate equations for $P(A \cap C)$

$$P(A \cap C) = P(A|C)P(C)$$
$$P(A \cap C) = P(C|A)P(A)$$
$$P(A \cap C) = P(A) - P(A \cap O)$$
$$P(A \cap C) = P(C) - P(M \cap C)$$

We will use the last equation but first we need $P(M \cap C)$ and $P(C)$

$$P(M \cap C) + P(M \cap O) = P(M)$$
$$P(M \cap C) = P(M) - P(M \cap O) = 0.2 - 0.014 = 0.186$$
$$P(C) = 1 - P(O) = 1 - 0.02 = 0.98$$
$$P(A \cap C) = P(C) - P(M \cap C) = 0.98 - 0.186 = 0.794$$

(c) What is the probability that a person is an adult given that they were counted in the census.

$$P(A|C) = \frac{P(A \cap C)}{P(C)} = \frac{0.794}{0.98} = 0.8102$$

Problem (3) (20 minutes - 20 points)
Consider the data that describes the concentration of product (mole/liter) when comparing three different companies feed-stocks. Each experiment was done with 12 replicates.

treat-ment	1	2	3	4	5	6	7	8	9	10	11	12
1	1.04	0.59	1.16	0.53	1.42	1.32	0.58	0.97	1.33	0.89	1.19	1.14
2	1.38	1.16	1.16	1.27	0.91	1.40	1.34	1.57	1.53	1.55	1.55	1.26
3	0.80	1.08	0.97	1.36	1.00	0.46	1.17	1.00	1.16	1.10	0.45	1.29

We run the data in the MATLAB code anova_1factor.m and obtain the following output:

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» anova_1factor(y,12,3,0.01)
  Ho: all treatments are equal
  Reject Ho if    6.37 >> f(    2,   33 )
  Hypothesis Rejected for 98 percent confidence interval (     6.37 >     5.31 )

  pvalue =   4.60e-003

  98 percent C.I. on the    1 treatment:  8.23e-001 <  1.01e+000 <  1.20e+000
  98 percent C.I. on the    2 treatment:  1.15e+000 <  1.34e+000 <  1.53e+000
  98 percent C.I. on the    3 treatment:  7.96e-001 <  9.87e-001 <  1.18e+000

  98 percent C.I. on the    1 -    2 treatment diff.: -5.96e-001 < -3.27e-001 < -5.71e-002
  98 percent C.I. on the    1 -    3 treatment diff.: -2.43e-001 <  2.67e-002 <  2.96e-001
  98 percent C.I. on the    2 -    3 treatment diff.:  8.37e-002 <  3.53e-001 <  6.23e-001
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(a) What is the physical reason for the rejection of the null hypothesis?
(b) Is there a statistically significant difference between treatments 1 & 2?
(c) Is there a statistically significant difference between treatments 1 & 3?
(d) Is there a statistically significant difference between treatments 2 & 3?
(e) If someone claims that using feed stock 3 instead of feed stock 1 will result in a 0.3 decrease in the amount of reactant leaving the reactor, do you believe him/her?

solution:
(a) What is the physical reason for the rejection of the null hypothesis?
treatment 2 gives a higher output than treatment 1 or 3
(b) Is there a statistically significant difference between treatments 1 & 2?
Yes. The mean of treatment 1 does not fall within the confidence interval for treatment 2.

(c) Is there a statistically significant difference between treatments 1 & 3?
No. The mean of treatment 1 falls within the confidence interval for treatment 3.
(d) Is there a statistically significant difference between treatments 2 & 3? Yes.
Yes. The mean of treatment 2 does not fall within the confidence interval for treatment 2.
(e) If someone claims that using feed stock 3 instead of feed stock 1 will result in a 0.3 decrease in the amount of reactant leaving the reactor, do you believe him/her?
No. 0.3 is not within the confidence interval for the difference of means between treatments 1 & 3.

Problem (4) (20 minutes - 20 points)
Perform one complete Newton-Raphson iteration on the system of equations:

$$y = \ln(x)$$

$$y^2 + \sqrt{x} = 4$$

Use (x,y) = (1,1) as your initial guess.

solution:

$$f_1(x, y) = \ln(x) - y = 0$$

$$f_2(x, y) = 4 - y^2 - \sqrt{x} = 0$$

$$\underline{\underline{J}} = \begin{bmatrix} \frac{1}{x} & -1 \\ -\frac{1}{2\sqrt{x}} & -2y \end{bmatrix} \quad \underline{R} = \begin{bmatrix} \ln(x) - y \\ 4 - y^2 - \sqrt{x} \end{bmatrix}$$

$$\underline{\underline{J}}(x = 1, y = 1) = \begin{bmatrix} 1 & -1 \\ -\frac{1}{2} & -2 \end{bmatrix}, \quad \underline{R}(x = 1, y = 1) = \begin{bmatrix} \ln(1) - 1 \\ 4 - 1 - 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\underline{\underline{J}}^{-1} = \frac{1}{\det(J)} \begin{bmatrix} j_{22} & -j_{12} \\ -j_{21} & j_{11} \end{bmatrix} = \frac{1}{-5/2} \begin{bmatrix} -2 & 1 \\ 1/2 & 1 \end{bmatrix} = \begin{bmatrix} 4/5 & -2/5 \\ -1/5 & -2/5 \end{bmatrix}$$

$$\underline{\delta x} = -\underline{\underline{J}}^{-1}\underline{R} = -\begin{bmatrix} 4/5 & -2/5 \\ -1/5 & -2/5 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1.6 \\ 0.6 \end{bmatrix}$$

$$\underline{x}^{(1)} = \underline{x}^{(0)} + \underline{\delta x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1.6 \\ 0.6 \end{bmatrix} = \begin{bmatrix} 2.6 \\ 2.6 \end{bmatrix}$$

There is no obvious reason this isn't a good initial guess. The functions can still be evaluated for the new values of x.

Problem (5) (20 minutes - 20 points)
You know the trapezoidal rule for n intervals, namely:

$$\frac{b-a}{2n}\left[f(a)+f(b)+2\sum_{i=2}^{n}f(x_i)\right]\approx\int_a^b f(x)dx \tag{5.1}$$

Show how two repeated applications of the trapezoidal rule can be used to approximate the double integral:

$$\int_c^d\int_a^b f(x,y)dxdy \tag{5.2}$$

Use the same same number of intervals for x and y. Provide a formula analogous to equation (5.1). Use the formula to evaluate the integral, using n=2.

$$\int_0^4\int_0^1 xe^y dxdy$$

solution:

$$\int_c^d\int_a^b f(x,y)dxdy = \int_c^d\left\{\int_a^b f(x,y)dx\right\}dy = \int_c^d\left\{\frac{b-a}{2n}\left[f(a,y)+f(b,y)+2\sum_{i=2}^{n}f(x_i,y)\right]\right\}dy = \int_c^d g(y)dy$$

where

$$g(y) = \left\{\frac{b-a}{2n}\left[f(a,y)+f(b,y)+2\sum_{i=2}^{n}f(x_i,y)\right]\right\}$$

Then, applying the trapezoidal rule again:

$$\int_c^d g(y)dy = \frac{d-c}{2n}\left[g(c)+g(d)+2\sum_{j=2}^{n}g(y_j)\right]$$

Substituting back in for g, we have

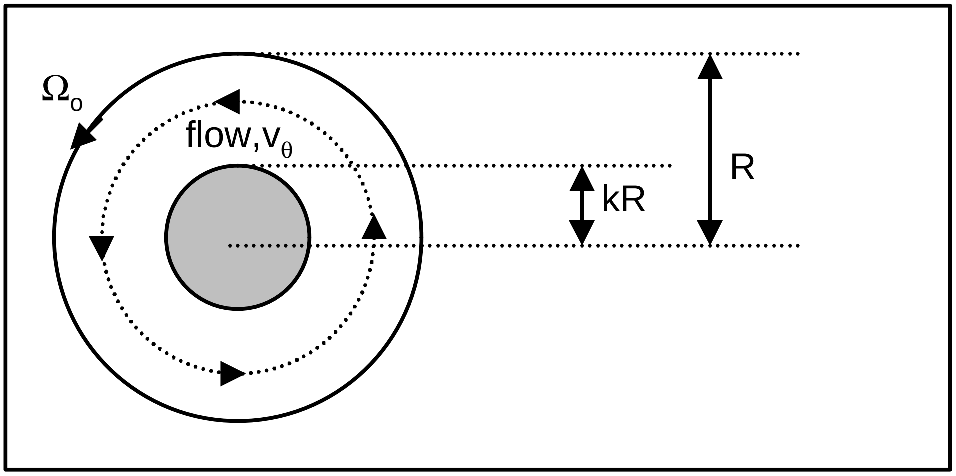
$$\int_c^d\int_a^b f(x,y)dxdy = \frac{d-c}{2n}\left[\begin{array}{l}\left\{\frac{b-a}{2n}\left[f(a,c)+f(b,c)+2\sum_{i=2}^{n}f(x_i,c)\right]\right\}\\ +\left\{\frac{b-a}{2n}\left[f(a,d)+f(b,d)+2\sum_{i=2}^{n}f(x_i,d)\right]\right\}\\ +2\sum_{j=2}^{n}\left\{\frac{b-a}{2n}\left[f(a,y_j)+f(b,y_j)+2\sum_{i=2}^{n}f(x_i,y_j)\right]\right\}\end{array}\right]$$

when a=0, b=1, c=0, d=4, n=2, and $f(x,y) = xe^y$, we have

$$\int_c^d \int_a^b f(x,y)dxdy = \frac{4}{4}\left[\begin{array}{l} \left\{\frac{1}{4}\left[0e^0 + 1e^0 + 2\cdot\frac{1}{2}e^0\right]\right\} + \left\{\frac{1}{4}\left[0e^4 + 1e^4 + 2\cdot\frac{1}{2}e^4\right]\right\} \\ + 2\left\{\frac{1}{4}\left[0e^2 + 1e^2 + 2\cdot\frac{1}{2}e^2\right]\right\} \end{array}\right]$$

$$\int_c^d \int_a^b f(x,y)dxdy = \frac{1}{2}\left(1 + e^4 + 2e^2\right) = 35.1881$$

Problem (6) (20 minutes - 20 points)
Consider the tangential annular flow of a Newtonian incompressible fluid between two vertical, coaxial cylinders. The inner cylinder has a radius kR and is immobile. The outer cylinder has a radius R and rotates with an angular velocity Ω_o, with units of radians/sec.



Starting with the equation of continuity and the equation of motion, and assuming (i) no flow in the radial and axial directions, (ii) symmetry in the angular direction, we arrive at the following equation:

$$\frac{d}{dr}\left(\frac{1}{r}\frac{d(rv_\theta)}{dr}\right) = 0$$

where r is the radial position and v_θ is the angular velocity, subject to the conditions:

$$v_\theta(r = kR) = 0$$
$$v_\theta(r = R) = \Omega_o$$

Your task is to describe the flow system. Describe how you would solve this problem numerically. You do not need to actually obtain a solution. In your answer, include the following points:

(a) identify the independent variable
(b) identify the dependent variable
(c) simplify the equation
(d) identify the type of equation (linear vs nonlinear)
(e) determine the order of the differential equation
(f) identify the conditions: (initial conditions vs boundary conditions)
(g) if necessary, transform a single n^{th}-order equation into a system of n first-order equations.
(h) name and describe the standard numerical algorithm needed to solve this problem
(i) predict the difficulty/ease of obtaining a solution with the method from (h)

solution:

(a) identify the independent variable: r
(b) identify the dependent variable: v_θ
(c) simplify the equation:

$$\frac{d}{dr}\left(\frac{1}{r}\frac{d(rv_\theta)}{dr}\right)=\frac{d}{dr}\left(\frac{1}{r}\left[r\frac{dv_\theta}{dr}+v_\theta\frac{dr}{dr}\right]\right)=\frac{d}{dr}\left(\frac{dv_\theta}{dr}+\frac{v_\theta}{r}\right)=0$$

$$\frac{d^2v_\theta}{dr^2}+\frac{1}{r}\frac{dv_\theta}{dr}-\frac{v_\theta}{r^2}=0 \text{ or } \frac{d^2v_\theta}{dr^2}=-\frac{1}{r}\frac{dv_\theta}{dr}+\frac{v_\theta}{r^2}$$

(d) identify the type of equation (linear vs nonlinear): The equation is linear in the independent unknown.

(e) determine the order of the differential equation: second order

(f) identify the conditions: (initial conditions vs boundary conditions)

boundary conditions on the angular velocity.

(g) if necessary, transform a single n^{th}-order equation into a system of n first-order equations.

$y_1 = v_\theta$ and $y_2 = \frac{dv_\theta}{dr}$ so the 2first-order odes become

$$\frac{dy_1}{dr}=y_2$$

$$\frac{dy_2}{dr}=-\frac{y_2}{r}+\frac{y_1}{r^2}$$

(h) name and describe the standard numerical algorithm needed to solve this problem

shooting method

(i) predict the difficulty/ease of obtaining a solution with the method from (h)

The shooting method is guaranteed to work because the equations are linear.