Midterm Examination Number Two
Administered: Wednesday, October 12, 1999

In all relevant problems: WRITE DOWN THE FORMULA YOU USE, BEFORE YOU USE IT.
ALL PROBLEMS ARE WORTH 2 POINTS.

Problem (1)

Consider flow down a circular pipe. The average velocity, $\bar{v}$, has a mean value of $\mu_{\bar{v}} = 0.131\frac{m}{s}$. The mean of the square of the average velocity is $\mu_{\bar{v}^2} = 0.482\frac{m}{s}$. The density, $\rho = 1000.0$ kg/m^3, the diameter, $D = 0.01$ m, and the viscosity, $\mu = 0.001\frac{kg}{m \cdot s}$ are constant.

(1.1) Find the standard deviation of the average velocity.

$$\sigma_{\bar{v}} = \sqrt{\sigma_{\bar{v}}^{\;2}} = \sqrt{\mu_{\bar{v}^2} - \mu_{\bar{v}}^{\;2}} = \sqrt{0.482 - 0.131^2} = 0.6818\frac{m}{s}$$

(1.2) Find the mean of the Reynolds number, $N_{Re} = \frac{D\bar{v}\rho}{\mu}$

$$\mu_{N_{Re}} = \mu_{\frac{D\bar{v}\rho}{\mu}} = \frac{D\rho}{\mu}\mu_{\bar{v}} = \frac{0.01 \cdot 1000}{0.001}0.131 = 1310$$

(1.3) Is the flow laminar, transitional, or turbulent?

Flow is laminar, based on the value of the Reynolds number.

(1.4) Find the standard deviation of the Reynolds number.

$$\sigma_{N_{Re}} = \sqrt{\sigma_{N_{Re}}^{\;2}} = \sqrt{\sigma_{\frac{D\bar{v}\rho}{\mu}}^{\;2}} = \sqrt{\left(\frac{D\rho}{\mu}\right)^2 \sigma_{\bar{v}}^{\;2}} = \frac{D\rho}{\mu}\sigma_{\bar{v}} = \frac{0.01 \cdot 1000}{0.001}0.6818 = 6818$$

(1.5) If we use the empirical correlation for the Nusselt Number: $N_{Nu} = 1.86\left(N_{Re}N_{Pr}\frac{D}{L}\right)^{\frac{1}{3}}$ where N_{Pr} is the Prandtl number and L is the length (both constants), would be expect the relative deviation of the Nusselt number (defined as $\frac{\sigma_{N_{Nu}}}{\mu_{N_{Nu}}}$) to be larger, the same as, or smaller than the relative deviation of the Reynolds number (defined as $\frac{\sigma_{N_{Re}}}{\mu_{N_{Re}}}$)? Why?

$$\mu_{N_{Nu}} \approx 1.86\left(\mu_{N_{Re}} N_{Pr} \frac{D}{L}\right)^{\frac{1}{3}} \quad \text{and} \quad \sigma_{N_{Nu}} \approx 1.86\left(\sigma_{N_{Re}} N_{Pr} \frac{D}{L}\right)^{\frac{1}{3}}$$

so

$$\frac{\sigma_{N_{Nu}}}{\mu_{N_{Nu}}} \approx \frac{1.86\left(\sigma_{N_{Re}} N_{Pr} \frac{D}{L}\right)^{\frac{1}{3}}}{1.86\left(\mu_{N_{Re}} N_{Pr} \frac{D}{L}\right)^{\frac{1}{3}}} = \left(\frac{\sigma_{N_{Re}}}{\mu_{N_{Re}}}\right)^{\frac{1}{3}}$$

Because the Reynolds number is a positive number, so its mean and standard deviation are positive numbers. Then we have that

$$\frac{\sigma_{N_{Nu}}}{\mu_{N_{Nu}}} \approx \left(\frac{\sigma_{N_{Re}}}{\mu_{N_{Re}}}\right)^{\frac{1}{3}} < \frac{\sigma_{N_{Re}}}{\mu_{N_{Re}}} \qquad \text{for} \qquad \frac{\sigma_{N_{Re}}}{\mu_{N_{Re}}} > 1$$

$$\frac{\sigma_{N_{Nu}}}{\mu_{N_{Nu}}} \approx \left(\frac{\sigma_{N_{Re}}}{\mu_{N_{Re}}}\right)^{\frac{1}{3}} > \frac{\sigma_{N_{Re}}}{\mu_{N_{Re}}} \qquad \text{for} \qquad \frac{\sigma_{N_{Re}}}{\mu_{N_{Re}}} < 1$$

Since in our case, we have

$$\frac{\sigma_{N_{Re}}}{\mu_{N_{Re}}} = \frac{6818}{1310} = 5.205 > 1$$

So $\frac{\sigma_{N_{Nu}}}{\mu_{N_{Nu}}}$ is less than $\frac{\sigma_{N_{Re}}}{\mu_{N_{Re}}}$.

Problem 2.

We are in the business of manufacturing injection-molded plastic fenders to automobile makers. We claim that our fenders will remain intact under head-on impact with a standard concrete pylon up to an average speed of 24 mph with a standard deviation of 3 mph. Our competitor claims that they have developed a new additive to their plastic which allows their bumpers to remain intact under the same conditions up to an average speed of 30 mph with a standard deviation of 5 mph. We don't believe this claim one bit. We test 12 fenders, half with our fenders and half with the competition's fenders. From this sample, we find that our bumpers do not fracture until 24.2 mph with a standard deviation of 2.9 mph. From the competition's sample, we find that their bumpers do not fracture until 30.1 mph with a standard deviation of 10 mph.

(a) Find a 98% confidence interval for the difference in the two companies' product's average life-times, assuming the claimed population standard deviations are believable.

(b) Does the claimed average life-time difference fall within this confidence interval?

(c) Find a 98% confidence interval for the difference in the two companies' product's average life-times, assuming the claimed population standard deviations are not believable.

(d) Does the claimed average life-time difference fall within this confidence interval?

(e) Does the test in (c) allow for the possibility that our product has a higher mean than the competition?

Solution:

$\mu_1 = 24$, $\sigma_1 = 3$, $\sigma_1{}^2 = 9$, $n_1 = 6$, $\bar{x}_1 = 24.2$, $s_1 = 2.9$, $s_1{}^2 = 8.41$
$\mu_2 = 30$, $\sigma_2 = 4$, $\sigma_2{}^2 = 16$, $n_2 = 6$, $\bar{x}_2 = 30.1$, $s_2 = 10$, $s_2{}^2 = 100$

(a)
$1-2\alpha = 0.98$, $\alpha = 0.01$, $z_\alpha = z_{0.01} = -2.33$, $z_{1-\alpha} = -z_\alpha = 2.33$

$$P\left[\left(\bar{X}_1 - \bar{X}_2\right) + z_\alpha\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} < \left(\mu_1 - \mu_2\right) < \left(\bar{X}_1 - \bar{X}_2\right) - z_\alpha\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}\right] = 1-2\alpha$$

$$P\left[(24.2-30.1) - 2.33\sqrt{\frac{9}{6} + \frac{16}{6}} < \left(\mu_1 - \mu_2\right) < (24.2-30.1) + 2.33\sqrt{\frac{9}{6} + \frac{16}{6}}\right] = 0.98$$

$$P\left[-10.65 < \left(\mu_1 - \mu_2\right) < -1.15\right] = 0.98$$

so the 98% confidence interval for the mean is

$$-10.65 < \left(\mu_1 - \mu_2\right) < -1.15$$

(b)
The claimed difference of population means $\mu_1 - \mu_2 = -6$ falls within this interval.

(c)
$1-2\alpha = 0.98$, $\alpha = 0.01$, $t_\alpha = t_{0.01} = -2.33$, $t_{1-\alpha} = -t_\alpha = 2.33$

$$v = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\left[\left(\frac{s_1^2}{n_1}\right)^2 \Big/ (n_1 - 1)\right] + \left[\left(\frac{s_2^2}{n_2}\right)^2 \Big/ (n_2 - 1)\right]} \text{ if } \sigma_1 \neq \sigma_2$$

$$v = \frac{\left(\frac{8.41}{6} + \frac{100}{6}\right)^2}{\left[\left(\frac{8.41}{6}\right)^2 \Big/ (6-1)\right] + \left[\left(\frac{100}{6}\right)^2 \Big/ (6-1)\right]} = 5.8351 \approx 6$$

$t_\alpha = t_{0.01} = -3.143$, $t_{1-\alpha} = -t_\alpha = 3.143$

The t value came from the Table A.4, of the t-PDF values.

$$P\left[(24.2-30.1) - 3.143\sqrt{\frac{8.41}{6} + \frac{100}{6}} < \left(\mu_1 - \mu_2\right) < (24.2-30.1) + 3.143\sqrt{\frac{8.41}{6} + \frac{100}{6}}\right] = 0.98$$

$$P\left[-19.2599 < \left(\mu_1 - \mu_2\right) < 7.4599\right] = 0.98$$

so the 98% confidence interval for the mean is

$$-19.3 < (\mu_1 - \mu_2) < 7.5$$

(d) Does the claimed average life-time difference fall within this confidence interval?
The claimed difference of population means $\mu_1 - \mu_2 = -6$ falls within this interval.

(e) Does the test in (c) allow for the possibility that our product has a higher mean than the competition?
Yes, it does because positive values of $\mu_1 - \mu_2$ are included in the confidence interval.

Problem 3.

Before Christmas one year, we place out an advent wreath with four candles. Each candle has an average lifetime of 4 hours. If we light the candles at 9:00 p.m., what is the probability that at least 3 of them are still burning by midnight?

Solution:

Use exponential PDF for each individual candle. Use binomial PDF for combination.

$$P(t > t_i) = \int_{t_i}^{\infty} f(t)dt = \int_{t_i}^{\infty} \frac{1}{\beta} e^{-\frac{t}{\beta}} dt = \int_{3}^{\infty} \frac{1}{4} e^{-\frac{t}{4}} dt = e^{-\frac{3}{4}} = 0.4724$$

Now use binomial with n=4 and p=0.4724

$$P(x \ge 3) = P(x = 3) + P(x = 4) = b(3;4,0.4724) + b(4;4,0.4724)$$

$$P(x \ge 3) = \binom{4}{3} 0.4724^3 (1 - 0.4724)^{4-3} + \binom{4}{4} 0.4724^4 (1 - 0.4724)^{4-4}$$

$$P(x \ge 3) = 0.2723$$

Problem 4.

Driving to school each morning, we encounter 6 traffic lights. Each traffic light stays green for 45 seconds, yellow for 5 seconds, and red for 50 seconds. Assuming that there is absolutely no synchronization among the streetlights and assuming that we don’t run yellow lights, find the probability that in a single morning, we hit 2 green lights, 2 yellow lights, and 2 red lights.

Solution.

Use the multinomial PDF.

$$P(\{X = x\}) = m(\{x\}; n, \{p\}, k) = \binom{n}{x_1, x_2 \ldots x_k} \prod_{i=1}^{k} p_i^{x_i}$$

$$P(\{X = x\}) = \binom{6}{2,2,2} (0.45)^2 (0.05)^2 (0.50)^2$$

$$P(\{X = x\}) = \frac{6!}{2!2!2!}(0.45)^2(0.05)^2(0.50)^2 = 0.0114$$

Problem 5.

For the situation in problem (4), what is the average time spent waiting at a light on a single morning? Assume your arrival time is random (i.e. use the continuous uniform PDF).

Definition of the average time.

$$\mu = E(X) = \sum_x xf(x)$$

Definition of the PDF as given by problem (4).

$$f(x) = \begin{cases} 0.45 & \text{green} \\ 0.05 & \text{yellow} \\ 0.50 & \text{red} \end{cases}$$

x is the random variable representing waiting at the light.

When we have a green light, x is 0 because we don't wait at a green light.

When we have a red light x is something between 0 and 50 seconds. I said to use the continuous uniform PDF for our arrival time so the average wait at a red light is 50/2.

When we hit a yellow light, we have to wait between 0 and 5 seconds on the yellow then we have to wait all of the red light.

$$x = \begin{cases} 0 & \text{seconds waiting on green} \\ 5/2 \text{ seconds waiting on yellow} + 50 & \text{seconds waiting on red} \\ 50/2 & \text{seconds waiting on red} \end{cases}$$

Plugging f(x) and x into the formula for the average, we have:

$$\mu = 0 \cdot 0.45 + (2.5 + 50) \cdot 0.05 + 25 \cdot 0.50 = 15.125 \text{ seconds}$$