Applied Statistics and Numerical Methods for Engineers
ChE 301, Fall 1998
Midterm Exam Number Two
Administered: Wednesday, October 13, 1998

ALL PROBLEMS ARE WORTH 2 POINTS. THE EXAM HAS 80 POINTS.

Do **10** of the 11 problems from Problem 1 to Problem 11.
Problem 1 to 11. Each problem is worth 6 points. For each problem:
(a) **Name the PDF** you choose to employ. **-** 2 points
(b) **Identify the numerical values for all parameters and variables**, which are arguments in the PDF. **-** 2 points.
(c) **Find the probability, expectation value, or statistic** requested. **-** 2 points.

Problem 1.

A plant produces a liquid waste-stream with, on average 6 ppm Chromium ion with standard deviation of 3 ppm. The current, local environmental limit is 10 ppm.

(c) What is the probability that a given sample reads less than the environmental limit?

Solution:

(a) Normal PDF.

$$f(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

(b) $\mu = 6$ and $\sigma = 3$ and $x \le 10$

(c) $z = \dfrac{x-\mu}{\sigma} = \dfrac{10-6}{3}$

$$P(X \le 10) = P\left(Z \le \frac{4}{3}\right) = 0.9082$$ From Table A.3

Problem 2.

An outdoor motion sensor is advertised as detecting 90% of human trespassers. During a demonstration test, 20 people walk past the motion sensor.

(c) What is the probability that the motion sensor detects 15 or fewer people.

Solution:

(a) Binomial PDF.

$$b(x;n,p) = \binom{n}{x} p^x q^{n-x}$$

(b) $p = 0.9$ and $q = 1 - p = 0.1$ and $n = 20$ and $x \le 15$

(c) $P(X \le 15) = B(15;20,0.9) = 0.0432$ From Table A.1

Problem 3.

An outdoor motion sensor is advertised as detecting 90% of human trespassers. During a demonstration test, people walk past the motion sensor.

(c) What is the probability that the motion sensor MISSES the first person on the 12^{th} pass?

Solution #1:

(a) geometric PDF.

$$g(x;p) = pq^{x-1} \quad \text{for } x = 1,2,3...$$

(b) $p = 0.1$ and $q = 1 - p = 0.9$ and $x = 12$

(c) $P(X = 12) = (0.1)(0.9)^{12-1} = 0.0314$

Solution #2:

(a) negative binomial PDF.

$$b^*(x;k,p) = \binom{x-1}{k-1} p^k q^{x-k} \quad \text{for } x = k, k+1, k+2...$$

(b) $p = 0.1$ and $q = 1 - p = 0.9$ and $x = 12$ and $k = 1$

(c) $P(X = 12) = \binom{12-1}{1-1}(0.1)(0.9)^{12-1} = 0.0314$

Problem 4.

An outdoor motion sensor is advertised as detecting 90% of human trespassers. During a demonstration test, people walk past the motion sensor.

(c) What is the probability that the motion sensor MISSES for the fifth time on the 20^{th} pass?

Solution:

(a) negative binomial PDF.

$$b^*(x;k,p) = \binom{x-1}{k-1} p^k q^{x-k} \quad \text{for } x = k, k+1, k+2...$$

(b) $p = 0.1$ and $q = 1 - p = 0.9$ and $x = 20$ and $k = 5$

(c) $P(X = 20) = \binom{20-1}{5-1}(0.1)^5(0.9)^{20-5} = 0.0080$

Problem 5.

A computer code is written to generate a real random-number between 2 and 99.

(c) What is the variance of the random number?

Solution:

(a) continuous uniform PDF.

$$f(x;A,B) = \begin{cases} \dfrac{1}{B-A} & \text{for } A \le x \le B \\ 0 & \text{otherwise} \end{cases}$$

(b) $B = 99$ and $A = 2$

(c) $\sigma^2 = \dfrac{(B-A)^2}{12} = \dfrac{(99-2)^2}{12} = 784.08$

Problem 6.

The University of Tennessee has contracts with the Personal Computer manufacturers: Gateway 2000, Dell, Compaq, and IBM. A computer lab is outfitted with 10 Gateway 2000's, 8 Dell's, 4 Compaq's, and 2 IBM's. One night, two computers are stolen from the lab. The thieves randomly selected the computers.

(c) What is the probability that one Dell and one IBM computer were stolen?

Solution:

(a) multivariate hypergeometric PDF.

$$h(\{x\};N,n,\{a\}) = \frac{\binom{a_1}{x_1}\binom{a_2}{x_2}\binom{a_3}{x_3}\cdots\binom{a_k}{x_k}}{\binom{N}{n}}$$

(b) $a_1 = 10$, $a_2 = 8$, $a_3 = 4$, $a_4 = 2$, $N = \sum a_i = 24$

$x_1 = 0$, $x_2 = 1$, $x_3 = 0$, $x_4 = 1$, $n = \sum x_i = 2$

(c) $$P(\{X\} = \{0,1,0,1\}) = \frac{\binom{10}{0}\binom{8}{1}\binom{4}{0}\binom{2}{1}}{\binom{24}{2}} = 0.0580$$

Problem 7.

An electrical engineering firm studies the neutral to full-load current ratio in the power-systems of personal computers across the nation. They find that 10% of the systems have high ratios, 30% have moderate ratios, 60% have low ratios. If an electrical engineer subsequently tests a sample of 10 computers is,

(c) What is the probability that the electrical engineer finds 1 high ratio, 3 moderate ratios, and 6 low ratios?

Solution:

(a) multinomial PDF.

$$m(\{x\};n,\{p\},k) = \binom{n}{x_1, x_2 \ldots x_k}\prod_{i=1}^{k} p_i^{x_i}$$

(b) $p_1 = 0.1$, $p_2 = 0.3$, $p_3 = 0.6$, $k = 3$

$x_1 = 1$, $x_2 = 3$, $x_3 = 6$, $n = 10$

(c) $$P(\{X\} = \{1,3,6\}) = \binom{10}{1,3,6}\prod_{i=1}^{3} p_i^{x_i} = 0.106$$

Problem 8.

While studying for an exam at 3 a.m., your street loses electrical power. Hastily you light three candles with mean life-times of 4 hours.

(c) What is the probability that at least one of the candles is still burning when the sun rises at 7 a.m.?

Solution:

(a) Binomial PDF and exponential PDF.

$$b(x;n,p) = \binom{n}{x} p^x q^{n-x} \text{ and } f_e(x;\beta) = \begin{cases} \frac{1}{\beta} e^{-x/\beta} & \text{for } x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

(b) For exponential: $\beta = 4$, $x \geq 4$

For binomial: $p = ?$ and $q = 1 - p$ and $n = 3$ and $x \geq 1$

(c) $p = P(t_i < t) = \int_{t_i}^{\infty} f_e(t;\beta)dt = \int_{t_i}^{\infty} \frac{1}{\beta} e^{-t/\beta} dt = e^{-t_i/\beta} = e^{-1} = 0.3679$

$$P(x \geq 1) = 1 - P(x \leq 0) = 1 - B(0;3,0.3679) = 1 - 0.2547 = 0.7453$$

Problem 9.

In the Washington DC metropolitan area, commuters can take the Orange, Yellow, or Blue lines of the subway system to cross the Potomac river from their homes in Virginia to Capitol Hill. For a sample of nine Virginia-DC commuters, we find that 2 rode the orange line, 3 rode the yellow line, and 4 rode the blue line.

(c) If we take 3 of these nine commuters to lunch in DC, what is the probability that we have lunch with 3 commuters from the blue line?

Solution:

(a) hypergeometric PDF.

$$h(x;N,n,k) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}} \quad \text{for } x = 0,1,2...n$$

(b) $k = 4$, $N = 9$, $n = 3$, $x = 3$

(c) $P(x = 3) = \frac{\binom{4}{3}\binom{9-4}{3-3}}{\binom{9}{3}} = 0.0476$

Problem 10.

In-depth research indicates that one out of every thousand bugs that fly into an electric bug-zapper live. If over the course of two months, 14,000 bugs fly into the electric bug-zapper,

(c) What is the probability that more than 20 bugs live?

Solution:

(a) Poisson PDF.

$$p(x;\lambda t) = \frac{e^{-\lambda t}(\lambda t)^x}{x!} \quad \text{for } x = 0,1,2...$$

(b) $\lambda = \frac{1}{1000}$ and $t = 14000$ and $x > 20$

(c) $P(x > 20) = 1 - P(x \leq 20) = 1 - p(20;14) = 1 - 0.9521 = 0.0479$ From Table A.2

Alternate Solution:

(a) binomial

(b) $p = 0.001$ and $q = 1 - p = 0.999$ and $n = 14000$ and $x < 20$

(c) $P(x > 20) = 1 - P(x \leq 20)$

We don't have a table for Binomial values with $n = 14000$, use normal approximation to the binomial:

$\mu = np = 14$, $\sigma = \sqrt{npq} = 3.740$, $z = \frac{x-\mu}{\sigma} = \frac{20-14}{3.740} = 1.604$

From Table A.3, $P(x < 1.604) = 0.9452$

so $P(x > 20) = 1 - P(x \le 20) = 1 - 0.9452 = 0.0548$

Problem 11.

Nocturnal research indicates that a bug flies into an electric bug-zapper every 30 seconds, on average.

(c) What is the probability that no bugs fly into the bug zapper over the course of one minute?

Solution:

(a) Exponential PDF.

$$f_e(x;\beta) = \begin{cases} \frac{1}{\beta} e^{-x/\beta} & \text{for } x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

(b) For exponential: $\beta = 30$, $x \ge 60$

(c) $p = P(t_i < t) = \int_{t_i}^{\infty} f_e(t;\beta)dt = \int_{t_i}^{\infty} \frac{1}{\beta} e^{-t/\beta}dt = e^{-t_i/\beta} = e^{-2} = 0.1353$

Alternate Solution:

(a) Poisson PDF.

$$p(x;\lambda t) = \frac{e^{-\lambda t}(\lambda t)^x}{x!} \quad \text{for } x = 0,1,2...$$

(b) $\lambda = \frac{1}{30}$ and $t = 60$ and $x = 0$

(c) $P(x = 0) = \frac{e^{-2}(\lambda t)^0}{0!} = e^{-2} = 0.1353$

Problem 12.

A manufacturer of fly-paper (strips of sweet, glue-covered paper that hang from the ceiling to catch flies) claims that her product permanently captures 92 percent of the flies who land on it, with a standard deviation of 3 percent. ($\mu_{fp} = 92$, $\sigma_{fp} = 3$) You sample 4 strips of fly paper.

(a) What is the probability that the mean effectiveness of the fly-paper sample is at least 89 percent?

(b) If our sample yields a mean $\overline{x}_{fp} = 89$, find a 95% confidence interval for the population mean, assuming the population variance you have been given is good.

(c) If our sample yields a mean $\overline{x}_{fp} = 89$ and standard deviation $s_{fp} = 5$, find a 95% confidence interval for the population mean, assuming the population variance is doubtful and not to be used.

(d) If our sample yields a mean $\overline{x}_{fp} = 89$ and standard deviation $s_{fp} = 5$, find a 95% confidence interval for the population variance, assuming the given value of the population variance is doubtful and not to be used.

(e) What PDFs did you use for parts (b) and (c) and (d)?

Solution:

(a) To estimate the mean, variance known, use the normal distribution.

$$z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}} = \frac{89 - 92}{3 / \sqrt{4}} = -2.0$$

$P(X \geq 89) = P(Z \geq -2.0) = 1 - P(Z < -2.0) = 1 - 0.0228 = 0.9772$ From Table A.3

(b) To estimate the mean, variance known, use the normal distribution.

$\alpha = 0.05$, $-z_{\alpha/2} = -z_{0.025} = -1.96$

$$P\left[\bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right] = 1 - \alpha$$

$$P[86.1 < \mu < 91.9] = 0.95$$

(c) To estimate the mean, variance unknown, use the t-distribution.

$v = n - 1 = 3$. First, find $t_{\alpha/2}$ for $\alpha = 0.05$ from table A.4, $t_{0.025} = 3.182$

$$P(\bar{X} - t_{\alpha/2}\frac{s}{\sqrt{n}} < \mu < \bar{X} + t_{\alpha/2}\frac{s}{\sqrt{n}}) = 1 - \alpha$$

$$P(81.0 < \mu < 97.0) = 0.95$$

(d) Use the chi-squared distribution to describe the distribution of the variance, when σ^2 is unknown.

$\alpha = 0.05$, $v = n - 1 = 3$

From table, A.5: $\chi^2_{\alpha/2} = 9.348$, $\chi^2_{1-\alpha/2} = 0.216$

$$P\left[\frac{(n-1)S^2}{\chi^2_{\alpha/2}} < \sigma^2 < \frac{(n-1)S^2}{\chi^2_{1-\alpha/2}}\right] = 1 - \alpha$$

$$P[8.023 < \sigma^2 < 347.2] = 0.95$$

(e) We used the normal, t-, and chi-squared distributions for parts (b), (c), and (d) respectively.

Problem 13.

You would like to know whether fly-paper ($\mu_{fp} = 92$, $\sigma_{fp} = 3$) is superior to bug-zappers ($\mu_{bz} = 95$, $\sigma_{bz} = 1$) in terms of percent effectiveness in removing flies. You use a bug-zapper for 9 nights. and fly-paper for 16 nights, discovering that $\bar{x}_{fp} = 93$ and $\bar{x}_{bz} = 94$.

(a) Find a 95% confidence interval that the difference between fly-paper and bug-zapper effective is really at least two percent.

(b) Does the stated population mean difference, $\mu_{fp} - \mu_{bz}$, fall within this confidence interval?

(c) If the statistics based on your sampling show, $s_{fp} = 1.5$ and $s_{bz} = 2.5$, find a 98% confidence interval that the population variance of the fly-paper effectiveness is really 3 times that of the variance of the bug-zapper.

(d) Does the stated population variance ratio, $\sigma_{fp} / \sigma_{bz}$ fall within this confidence interval?

(e) What PDFs did you use for parts (a) and (c)?

Solution:

(a) To estimate the difference of means with variance known, use the normal distribution. $\alpha = 0.05$, $-z_{\alpha/2} = -z_{0.025} = -1.96$ From table A.3

$$P\left[\left(\overline{X}_1 - \overline{X}_2\right) - z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} < (\mu_1 - \mu_2) < \left(\overline{X}_1 - \overline{X}_2\right) + z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}\right] = 1 - \alpha$$

$$P\left[-1 - 1.96\sqrt{\frac{3^2}{9} + \frac{1^2}{16}} < (\mu_1 - \mu_2) < -1 + 1.96\sqrt{\frac{3^2}{9} + \frac{1^2}{16}}\right] = 0.95$$

$$P[-2.61 < (\mu_1 - \mu_2) < 0.61] = 0.95$$

(b) No, the sampling data $\mu_{fp} - \mu_{bz} = -3$ does not fall within this confidence interval.

(c) Use the F-distribution to describe the distribution of the ratio of two variances.

$v_{fp} = 8$, $v_{bz} = 15$, $\alpha = 0.01$, $f_{\alpha/2}(v_1, v_2) = 4.00$ from table A.6

$f_{\alpha/2}(v_2, v_1) = 3.22$

$$P\left[\frac{S_1^2}{S_2^2}\frac{1}{f_{\alpha/2}(v_1, v_2)} < \frac{\sigma_1^2}{\sigma_2^2} < \frac{S_1^2}{S_2^2} f_{\alpha/2}(v_2, v_1)\right] = 1 - \alpha$$

$$P\left[0.09 < \frac{\sigma_1^2}{\sigma_2^2} < 1.16\right] = 0.98$$

(d) No, the ratio of variances that we were given, $\frac{\sigma_{fp}^2}{\sigma_{bz}^2} = 3$, does not fall into this confidence interval.

(e) We used the normal and F-distributions for part (a) and (c) respectively.