ChE/MSE 301
Applied Statistical and Numerical Methods for Engineers
Final Exam
Fall Semester, 2004
Instructor: David Keffer
Administered: 8:00-10:00 am, Thursday December 9, 2004

Problem (1). (10 points)

Consider the nonlinear ordinary differential equation initial value problem

$$\frac{dy}{dx} = 3\frac{x}{y^2} \qquad (2.1)$$

subject to the initial condition $y(x_o = 3) = y_o = 3$

Solve the ODE as it is given in equation (2.1) using the Euler Method. Use a step size of Δx=1 and report the solution at x=5.

Solution:

Solve the ODE as it is given in equation (2.1) using the Euler Method. Use a step size of Δx=1 and report the solution at x=5.

The Euler method is given by the equation

$$y_{i+1} = y_i + \Delta x \left(\frac{dy}{dx} \right)_{x=x_i}$$

Applying this formula twice and equation (2.1) and the initial condition yields

i	x	y(x)	dy/dx
0	3	3	1
1	4	4	¾
2	5	4 ¾	

Therefore, $y(x=5)$ = 4 ¾ = 4.75.

Problem (2) (10 points)

We are in the business of producing batteries. Our historical data indicates that on average our batteries last 4 years with a standard deviation of 3 months. We are concerned with the warranty on the batteries.

(a) If we warranty the battery for 42 months, what is the fraction of batteries that we could expect to replace?

(b) If we only want to replace 2% of the batteries, how long should our warranty be for?

(c) What PDF did you use to solve this problem?

Solution:

(a) If we warranty the battery for 42 months, what is the fraction of batteries that we could expect to replace?

This problem can be classified as given z find p.

$$z = \frac{x - \mu}{\sigma} = \frac{42 - 48}{3} = -2$$

$$p(x < 42) = p(z < -2) = 0.0228$$

The standard normal distribution value was obtained from appendix A.3 of WMM.
We can expect to replace 2.28% of the batteries if our warranty is for 42 months.

(b) If we only want to replace 2% of the batteries, how long should our warranty be for?

This problem can be classified as given p find z.

$$p(z < z_{lo}) = 0.02$$

From table A.3 in WMM, we find that $z_{lo} = -2.055$

$$z = \frac{x - \mu}{\sigma} = -2.055$$

$$x = \mu + \sigma z = 48 + 3(-2.055) = 41.835$$

We should warranty the batteries for 41.835 months.

(c) What PDF did you use to solve this problem?

I used the Normal PDF, because the random variable is continuous and all I was given was a mean and a standard deviation.

Problem (3) (10 points)

We are developing a process where the quality of the feedstock is important. Poor quality feedstock can result in unacceptable product. A vendor for the feedstock provides us with 18 samples. He *claims* that the population mean purity of the feed stock is 0.70 and *claims* that the

population standard deviation is 0.002. We run the samples through our own lab and find a sample mean purity of 0.711 with a sample standard deviation of 0.008. Based on this information, answer the following questions.
(a) What PDF is appropriate for determining a confidence interval on the variance?
(b) Find the lower limit on a 96% confidence interval on the variance.
(c) Find the upper limit on a 96% confidence interval on the variance.
(d) Is the vendor's claim legitimate?
(e) If our maximum allowable standard deviation is 0.010, can we be 96% confident that the vendor's feedstock is adequate?

Solution:

(a) What PDF is appropriate for determining a confidence interval on the variance?
Chi-squared distribution for the confidence interval on the variance.

(b) Find the lower limit on a 96% confidence interval on the variance.
(c) Find the upper limit on a 96% confidence interval on the variance.

$$\alpha = \frac{1-\text{C.I.}}{2} = 0.02 \qquad v = n-1 = 17$$

$$\chi^2_{\alpha} = 30.995 \text{ from table A.5} \qquad \chi^2_{1-\alpha} = 7.255 \text{ from table A.5}$$

$$s^2 = 0.008^2 = 6.4\cdot 10^{-5}$$

$$P\left[\frac{(n-1)s^2}{\chi^2_{\alpha}} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{1-\alpha}}\right] = 1-2\alpha$$

$$P\left[\frac{(18-1)0.008^2}{30.995} < \sigma^2 < \frac{(18-1)0.008^2}{7.255}\right] = 0.96$$

$$P\left[3.5102\cdot 10^{-5} < \sigma^2 < 1.4997\cdot 10^{-4}\right] = 0.96$$

(d) Is the vendor's claim legitimate?
The vendor claimed that $\sigma^2 = 0.002^2 = 4.0\cdot 10^{-6}$. Since this value does not fall within our confidence interval, his claim is not legitimate.

(e) If our maximum allowable standard deviation is 0.010, can we be 96% confident that the vendor's feedstock is adequate?
Our maximum allowable deviation is $s^2 = 0.010^2 = 1.0\cdot 10^{-4}$. This value lies within the confidence interval. Since this value is our maximum acceptable value, there are points within this confidence interval that we cannot accept. Therefore, we cannot be confident that the vendor's feedstock is adequate.

Problem 4. (10 points)

Consider an nxn matrix, $\underline{\underline{J}}$, with rank = n. Indicate which of any of the following statements are true.

(a) The inverse of $\underline{\underline{J}}$ exists.

(b) At least 2 rows of $\underline{\underline{J}}$ are linearly dependent.

(c) The determinant of $\underline{\underline{J}}$ is non-zero.

(d) There is a unique solution to $\underline{\underline{J}}\underline{x} = \underline{b}$ for any real nx1 vector, $\underline{b}$.

(e) The reduced row echelon form of $\underline{\underline{J}}$ will have one row completely filled with zeroes.

Solution:

(a) True
(b) False
(c) True
(d) True
(e) False

Problem 5. (10 points)
We want to use the following equation to fit some vapor pressure data.

$$P^{vap} = \exp\left(\frac{A}{B+T}\right) \tag{4}$$

where T is temperature and A and B are fitting constants. We have two pieces of data: the vapor pressure at 300 K is 1.1 atm and the vapor pressure at 320 K is 1.7 atm. Given this experimental data find the best values of A and B.

Solution:

We have two unknowns A and B. Our life would be much simpler if we can rearrange the problem so that it is linear in A and B. Let's try.

$$(B+T)\ln\left(P^{vap}\right) - A = 0$$

There we have it. The equation is linear in A and B. We have a system of two linear equations and two unknowns.

$$f_1(A,B) = (B+T_1)\ln\left(P_1^{vap}\right) - A$$
$$f_2(A,B) = (B+T_2)\ln\left(P_2^{vap}\right) - A$$

We can write this in matrix notation as

$$\underline{\underline{\mathbf{J}}}\underline{\mathbf{x}} = \underline{\mathbf{R}}$$

where

$$\underline{\underline{\mathbf{J}}} = \begin{bmatrix} -1 & \ln\left(P_1^{vap}\right) \\ -1 & \ln\left(P_2^{vap}\right) \end{bmatrix}, \ \underline{\mathbf{x}} = \begin{bmatrix} A \\ B \end{bmatrix}, \text{ and } \underline{\mathbf{R}} = \begin{bmatrix} -T_1 \ln\left(P_1^{vap}\right) \\ -T_2 \ln\left(P_2^{vap}\right) \end{bmatrix}$$

We need the determinant and inverse

$$\det\left(\underline{\underline{\mathbf{J}}}\right) = -\ln\left(P_2^{vap}\right) + \ln\left(P_1^{vap}\right) = \ln\left(\frac{P_1^{vap}}{P_2^{vap}}\right)$$

$$\underline{\underline{\mathbf{J}}}^{-1} = \frac{1}{\det\left(\underline{\underline{\mathbf{J}}}\right)} \begin{bmatrix} j_{22} & -j_{12} \\ -j_{21} & j_{11} \end{bmatrix} = \frac{1}{\ln\left(\frac{P_1^{vap}}{P_2^{vap}}\right)} \begin{bmatrix} \ln\left(P_2^{vap}\right) & -\ln\left(P_1^{vap}\right) \\ 1 & -1 \end{bmatrix}$$

The solution is given by

$$\underline{\mathbf{x}} = \underline{\underline{\mathbf{J}}}^{-1}\underline{\mathbf{R}} = \frac{1}{\ln\left(\frac{P_1^{vap}}{P_2^{vap}}\right)} \begin{bmatrix} \ln\left(P_2^{vap}\right) & -\ln\left(P_1^{vap}\right) \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -T_1 \ln\left(P_1^{vap}\right) \\ -T_2 \ln\left(P_2^{vap}\right) \end{bmatrix} = \frac{1}{\ln\left(\frac{P_1^{vap}}{P_2^{vap}}\right)} \begin{bmatrix} \left(T_2 - T_1\right)\ln\left(P_1^{vap}\right)\ln\left(P_2^{vap}\right) \\ -T_1 \ln\left(P_1^{vap}\right) + T_2 \ln\left(P_2^{vap}\right) \end{bmatrix}$$

We have two pieces of data: the vapor pressure at 300 K is 1.1 atm and the vapor pressure at 320 K is 1.7 atm.

$$\underline{\mathbf{x}} = \frac{1}{\ln\left(\frac{1.1}{1.7}\right)} \begin{bmatrix} 20\ln(1.1)\ln(1.7) \\ -300\ln(1.1) + 320\ln(1.7) \end{bmatrix} \approx -2.2972 \begin{bmatrix} 1.0115 \\ 141.21 \end{bmatrix} = \begin{bmatrix} -2.3236 \\ -324.38 \end{bmatrix}$$