Final Exam
Administered: 10:15 am -12:15 pm, Monday, December 9
42 points

For each problem part: 0 points if not attempted or no work shown,
1 point for partial credit, if work is shown,
2 points for correct numerical value of solution

Problem (1) (10 points)
Perform one complete Newton-Raphson iteration on the system of equations:

$$x - 5y = -2 \qquad\qquad xy^2 = 3$$

Use (x,y) = (2,2) as your initial guess.
Along the way, present the Jacobian, Residual, determinant, inverse, and new estimate of [x,y].

solution:

$$f_1(x,y) = x - 5y + 2 = 0 \qquad\qquad f_2(x,y) = xy^2 - 3 = 0$$

$$\underline{\underline{J}} = \begin{bmatrix} 1 & -5 \\ y^2 & 2xy \end{bmatrix} \qquad\qquad \underline{R} = \begin{bmatrix} x - 5y + 2 \\ xy^2 - 3 \end{bmatrix}$$

$$\underline{\underline{J}}(x=2, y=2) = \begin{bmatrix} 1 & -5 \\ 4 & 8 \end{bmatrix} \qquad \underline{R}(x=2, y=2) = \begin{bmatrix} x - 5y + 2 \\ xy^2 - 3 \end{bmatrix} = \begin{bmatrix} 2 - 10 + 2 \\ 8 - 3 \end{bmatrix} = \begin{bmatrix} -6 \\ 5 \end{bmatrix}$$

$$\det(J) = j_{11}j_{22} - j_{21}j_{12} = (1)(8) - (4)(-5) = 28$$

$$\underline{\underline{J}}^{-1} = \frac{1}{\det(J)}\begin{bmatrix} j_{22} & -j_{12} \\ -j_{21} & j_{11} \end{bmatrix} = \frac{1}{28}\begin{bmatrix} 8 & 5 \\ -4 & 1 \end{bmatrix}$$

$$\underline{\delta x} = -\underline{\underline{J}}^{-1}\underline{R} = -\frac{1}{28}\begin{bmatrix} 8 & 5 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} -6 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{23}{28} \\ -\frac{29}{28} \end{bmatrix}$$

$$\underline{x}^{(1)} = \underline{x}^{(0)} + \underline{\delta x} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} \frac{23}{28} \\ -\frac{29}{28} \end{bmatrix} = \begin{bmatrix} \frac{79}{28} \\ \frac{27}{28} \end{bmatrix} \approx \begin{bmatrix} 2.821 \\ 0.964 \end{bmatrix}$$

Problem (2). (8 points)

Consider the nonlinear ordinary differential equation initial value problem

$$\frac{dy}{dx} = \frac{x}{y^2} \tag{2.1}$$

subject to the initial condition $y(x_o = 2) = y_o = 2$

Solve the ODE as it is given in equation (2.1) using the Euler Method. Use a step size of Δx=1 and report the solution at x=4.

Solution:

Solve the ODE as it is given in equation (2.1) using the Euler Method. Use a step size of Δx=1 and report the solution at x=4.

The Euler method is given by the equation

$$y_{i+1} = y_i + \Delta x \left(\frac{dy}{dx} \right)_{x=x_i}$$

Applying this formula twice and equation (2.1) and the initial condition yields

i	x	y(x)	dy/dx
0	2	2	½
1	3	2 ½	12/25
2	4	149/50	

Therefore, y(x=4) = 149/50=2.98.

Problem (3) (6 points)

The production capacity of a plant in a given week depend upon a variety of uncontrollable factors including quality of feedstock, ambient temperature and humidity, shift personnel, and other unidentified factors. As a result a particular plant has a historical average of 1500 tons of product per week with a standard deviation of 200 tons.

(a) What is the probability in any given week that the plant manufactures less than 1000 tons of product?
(b) 95% of the weeks produce at least what amount of product?
(c) What PDF did you use to solve this problem?

Solution:

(a) What is the probability in any given week that the plant manufactures less than 1000 tons of product?

$$z = \frac{x-\mu}{\sigma} = \frac{1000-1500}{200} = -2.5$$

$$p(x < 1000) = p(z < -2.5) = 0.0062$$

The standard normal distribution value was obtained from appendix A.3 of WMM.

(b) 95% of the weeks produce at least what amount of product?

$$p(z > z_{lo}) = 0.95$$

$$p(z < z_{lo}) = 1 - 0.95 = 0.05$$

From table A.3 in WMM, we find that $z_{lo} = -1.645$

$$z = \frac{x-\mu}{\sigma} = -1.645$$

$$x = \mu + \sigma z = 1500 + 200(-1.645) = 1171$$

The process manufactures at least 1171 tons of product 95% of the time.

(c) What PDF did you use to solve this problem?

I used the Normal PDF, because the random variable is continuous and all I was given was a mean and a standard deviation.

Problem 4. (8 points)

A manufacturer of automobile batteries claims that their batteries last on average 8 years. The motorpool of your company buy twelve batteries. They last on average 5 years with a sample standard deviation of 1 year. Construct a 95% confidence interval on the mean battery life.

(a) What is the appropriate Probability Density Function (PDF) to use in this problem?
(b) What is the lower limit on a 98% confidence interval for the mean battery life?
(c) What is the upper limit on a 98% confidence interval for the mean battery life?
(d) Is the manufacturer's claim reasonable? Justify your answer.

Solution:

(a) What is the appropriate Probability Density Function (PDF) to use in this problem?

The correct PDF to use is the t-distribution because this is a problem concerning the distribution of the sample mean when the population variance is unknown.

(b) What is the lower limit on a 98% confidence interval for the mean battery life?
(c) What is the upper limit on a 98% confidence interval for the mean battery life?

To estimate the mean, variance unknown, use the t-distribution.

$v = n - 1 = 11$. First, find $t_{\alpha/2}$ for $\alpha = 0.02$ from table A.4, $t_{0.01}(v = 11) = 2.718$

$$P(\overline{X} - t_{\alpha/2}\frac{s}{\sqrt{n}} < \mu < \overline{X} + t_{\alpha/2}\frac{s}{\sqrt{n}}) = 1 - \alpha$$

$$P(5 - (2.718)\frac{1}{\sqrt{12}} < \mu < 5 + (2.718)\frac{1}{\sqrt{12}}) = 0.98$$

$$P(4.215 < \mu < 5.785) = 0.98$$

(d) Is the manufacturer's claim reasonable? Justify your answer.

The claim of 8 years is not reasonable because it does not fall within the 98% confidence interval.

Problem (5) (6 points)
According to the Sourcebook of Criminal Justice Statistics, in 2001 88.4% of the entire United States prison population was male. 43.0% of the entire prison population was white. Given that an inmate was white, 80% of those prisoners were male. Answer the following questions.

(a) Given that a prisoner is a male what is the probability that they are white?
(b) What is the probability that a prisoner is a female and is not white?
(c) What is the probability that a prisoner is a female given that they were not white.

solution:
The solution space is divided into four groups as shown below:

Male $\cap$ White	Female $\cap$ White
Male $\cap$ Not White	Female $\cap$ Not White

The Given Information is as follows:

$$P(M) = 0.884$$

$$P(W) = 0.430$$

$$P(M|W) = 0.800$$

(a) Given that a prisoner is a male what is the probability that they are white?

$$P(M|W) = 0.800 = \frac{P(W \cap M)}{P(W)} = \frac{P(W \cap M)}{0.430}$$
$$P(W \cap M) = 0.800 \cdot 0.430 = 0.344$$
$$P(W|M) = \frac{P(W \cap M)}{P(M)} = \frac{0.344}{0.884} = 0.389$$

(b) What is the probability that a prisoner is a female and is not white?

All four of the following are legitimate equations for $P(F \cap N)$

$$P(F \cap N) = P(F|N)P(N)$$
$$P(F \cap N) = P(N|F)P(F)$$
$$P(F \cap N) = P(F) - P(F \cap W)$$
$$P(F \cap N) = P(N - P(M \cap N)$$

We will use the last equation but first we need $P(M \cap N)$ and $P(N)$

$$P(M \cap N) + P(M \cap W) = P(M)$$
$$P(M \cap N) = P(M) - P(M \cap W) = 0.884 - 0.344 = 0.540$$
$$P(N) = 1 - P(W) = 1 - 0.430 = 0.570$$
$$P(F \cap N) = P(N) - P(M \cap N) = 0.570 - 0.540 = 0.030$$

(c) What is the probability that a prisoner is a female given that they were not white.

$$P(F|N) = \frac{P(F \cap N)}{P(N)} = \frac{0.030}{0.570} = 0.053$$

Problem (6) (4 points)

Consider the integral:

$$I = \int_{210}^{476} (57x + 99) dx$$

(a) Use the trapezoidal rule to numerically evaluate the following integral. Use as many intervals as you think is necessary to generate an accurate solution. (You must show your work. Do not provide an analytical solution to the integral. Analytical solutions will be given no credit.)
(b) Justify your choice of the number of intervals in the trapezoidal method.

Solution:

(a) Use the trapezoidal rule to numerically evaluate the following integral. Use as many intervals as you think is necessary to generate an accurate solution.

$$I = \int_a^b f(x)dx \approx \frac{b-a}{2n}\left[f(a) + f(b) + 2\sum_{i=2}^{n} f(x_i)\right]$$

where n is the number of intervals used in the trapezoidal approximation.

I am going to use one interval, so I have

$$I = \int_a^b f(x)dx \approx \frac{b-a}{2n}[f(a) + f(b)] = \frac{476-210}{2}[12069 + 27231] = 5{,}226{,}900$$

(b) Justify your choice of the number of intervals in the trapezoidal method.

You only need one interval to integrate the function exactly, since the integrand is linear.