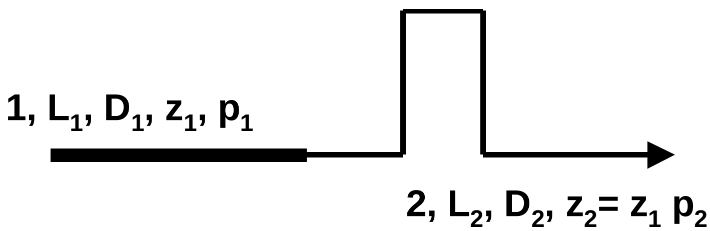
**Examination Number Two**
**Administered: Friday, March 5, 1999**

**Problem 1. Friction and mechanical energy balance**

Consider the following flow system:




where the fluid is water. Use a density, $\rho = 1000.0\frac{kg}{m^3}$ and a viscosity, $\mu = 1.0cp$. Assume steady state. The diameters of the lines are: $D_1 = 0.076m$ and $D_2 = 0.051m$. The lengths of the lines are: $L_1 = 10.0m$ and $L_2 = 30.0m$. The elevations of the lines are: $z_1 = 0.0m$ and $z_2 = z_1$. The volumetric flow rate feeding into the pump is $q_1 = 0.002\frac{m^3}{s}$.

(a) List and calculate all frictional terms in the mechanical energy balance. Calculate the total frictional loss. State all answers in [J/kg].

For your convenience, use the pre-calculated values:

$$\bar{v}_1 = \frac{q}{A_1} = 0.44\frac{m}{s} \qquad \bar{v}_2 = \frac{q}{A_2} = 0.98\frac{m}{s}$$

$$N_{Re,1} = \frac{D_1\bar{v}_1\rho}{\mu} = 33400 \qquad N_{Re,2} = \frac{D_2\bar{v}_2\rho}{\mu} = 50000$$

$$\frac{\varepsilon}{D_1} = 0.0006 \qquad \frac{\varepsilon}{D_2} = 0.0009$$

**Solution:**

Friction terms:

(1) skin friction in 10 m of 0.076 m pipe
from table on page 88, $f = 0.006$

$$F_f = \frac{4fL}{D}\frac{\bar{v}^2}{2} = 0.31\frac{J}{kg}$$ (2 pts)

(2) contraction from 0.076 to 0.051 m pipe

$$K_c = 0.55\left(1 - \frac{A_2}{A_1}\right) = 0.18,\ h_{f,c} = K_c \frac{\overline{v}^2}{2} = 0.15 \frac{J}{kg}$$ (2 pts)

(3) skin friction in 30 m of 0.051 m pipe

from table on page 88, $f = 0.007$

$$F_f = \frac{4fL}{D}\frac{\overline{v}^2}{2} = 7.91\frac{J}{kg}$$ (2 pts)

(4) 4 ninety degree elbows

$$4h_{f,L} = 4K_L \frac{\overline{v}^2}{2} = 4(0.75)\frac{\overline{v}^2}{2} = 1.44\frac{J}{kg}$$ (2 pts)

$$\sum F = 9.81\frac{J}{kg}$$ (2 pts)

(b) Calculate the pressure drop in the system. State your answer in [Pa].

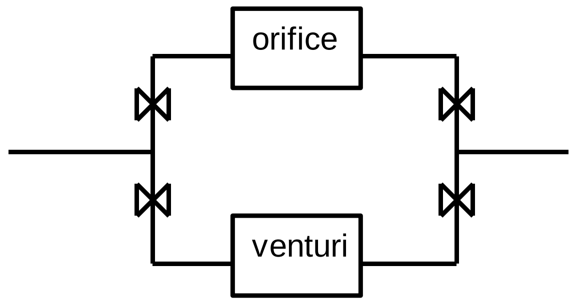
$$0 = \frac{\Delta p}{\rho} + \frac{\Delta v^2}{2\alpha} + g\Delta z + \hat{\dot{W}}_s + \sum \hat{F}$$ (2 pts)

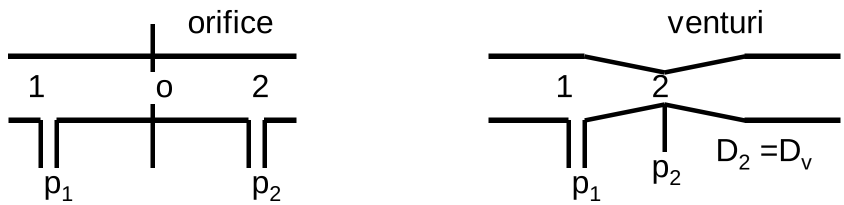
$$0 = \frac{\Delta p}{1000} + \frac{0.98^2 - 0.44^2}{2(1)} + 0 + 0 + 9.81\frac{J}{kg}$$

$$\Delta p = -10193 \text{ Pa}$$ (2 pts)

**Problem 2. Flow Measurement**

Consider the following flow system, designed to measure flow with either an orifice or a venturi meter, depending on whether the valves at the top or bottom are open. The pressure loss across either meter can be divided into two terms: a permanent pressure loss and a recoverable pressure loss, where some of the mechanical energy lost to the formation of vortices is reclaimed when the vortices dissipate downstream of the meters. The fraction of measured pressure loss that is permanent is approximately 10% for the venturi meter and 73% for the orifice. Find the orifice diameter that will give the same permanent pressure loss as the venturi throat diameter. Express your answer as $D_o = f(D_v, C_o, C_v, D_1)$.




(Hint: remember $\dot{m} = \rho \bar{v} A$ so $\bar{v} = \frac{\dot{m}}{\rho A} = \frac{4\dot{m}}{\rho \pi D^2}$)

**Solution:**

We know the formulae for venturi and orifice meters:

$$v_2 = \frac{C_v}{\sqrt{1-(D_2/D_1)^4}}\sqrt{\frac{2(p_2-p_1)}{\rho}} = \frac{4\dot{m}}{\rho \pi D_2^2} \qquad \text{(2 pts)}$$

$$v_o = \frac{C_o}{\sqrt{1-(D_o/D_1)^4}}\sqrt{\frac{2(p_1-p_2)}{\rho}} = \frac{4\dot{m}}{\rho \pi D_o^2} \qquad \text{(2 pts)}$$

We also know that the problem asks us to equate permanent pressure losses of the two meters.

$$0.73(p_2-p_1)_{orifice} = 0.10(p_2-p_1)_{venturi} \qquad \text{(2 pts)}$$

Rearrange the top two equations for pressure drop.

$$(p_2-p_1)_{venturi} = \frac{\rho}{2}\left[\frac{4\dot{m}\sqrt{1-(D_2/D_1)^4}}{\rho \pi D_2^2 C_v}\right]^2$$

$$(p_2-p_1)_{orifice} = \frac{\rho}{2}\left[\frac{4\dot{m}\sqrt{1-(D_o/D_1)^4}}{\rho \pi D_o^2 C_o}\right]^2$$

Substitute into the equation that equates permanent pressure losses:

$$0.73\frac{\rho}{2}\left[\frac{4\dot{m}\sqrt{1-(D_o/D_1)^4}}{\rho\pi D_o^2 C_o}\right]^2 = 0.10\frac{\rho}{2}\left[\frac{4\dot{m}\sqrt{1-(D_2/D_1)^4}}{\rho\pi D_2^2 C_v}\right]^2$$

$$0.73\frac{1-(D_o/D_1)^4}{D_o^4 C_o^2} = 0.10\frac{1-(D_2/D_1)^4}{D_2^4 C_v^2}$$

Rearrange for $D_o = f(D_v, C_o, C_v, D_1)$ where $D_v = D_2$

$$D_o = \left[\frac{1}{D_1^{-4} + \frac{0.10}{0.73}\frac{C_o^2}{C_v^2}\left(D_v^{-4} - D_1^{-4}\right)}\right]^{1/4}$$ (2 pts)

$D_o$ is larger than $D_v$. (2 pts)

**Problem 3. Navier-Stokes Equation**

An incompressible, Newtonian fluid flows down the side of a vertically upright steel plate. Let x designate the vertical dimension, y is a lateral dimension perpendicular to the face of the plate, and z is a lateral dimension parallel to the face of the plate. Consider one-dimensional flow down the plate where the fluid film has a thickness in the y-direction of $y_o$.

**Solution:**

(a) The Navier-Stokes equation can be written for the x, y, and z component of momentum. Which one component is of interest in this one-dimensional-flow problem? Why?

The x-component is of interest because the velocity in the x-direction is the only component that is non-zero. There is no flow in the y or z dimension.

(2 pts for right answer, 2 pts for explanation)

(b) For the component you selected in part (a) write out the full Navier-Stokes equation (don’t use substantial derivatives).

$$\rho\left(\frac{\partial v_x}{\partial t} + \frac{\partial v_x}{\partial x}v_x + \frac{\partial v_x}{\partial y}v_y + \frac{\partial v_x}{\partial z}v_z\right) = \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right) - \frac{\partial p}{\partial x} + \rho g_x$$

This is equation 3.7-36 on page 174 of Geankoplis. We select this equation because

(1) We are in a system where using rectangular coordinates makes sense.
(2) Only our x-component of velocity is non-zero.
(3) We have an incompressible Newtonian fluid.

(2 pts for right answer)

(c) For the equation you wrote in part (b), cross-out all negligible terms. Explain your reasoning in deleting terms. Is the resulting equation an ordinary differential equation or a partial differential equation?

$$\rho\left(\frac{\partial v_x}{\partial t}+\frac{\partial v_x}{\partial x}v_x+\frac{\partial v_x}{\partial y}v_y+\frac{\partial v_x}{\partial z}v_z\right)=\mu\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2}\right)-\frac{\partial p}{\partial x}+\rho g_x$$

Right hand side: First term drops out. Steady-state assumption.
Second term drops out. x-component of velocity does not vary in the x direction.
Third term drops out. y-component of velocity is zero.
Fourth term drops out. z-component of velocity is zero.

Left hand side: First term drops out. x-component of velocity does not vary in the x direction.
Second term does not drop out.
Third term drops out. x-component of velocity does not vary in the z direction
Fourth term drops out. No pressure gradient in a free-falling fluid.
Fifth term remains. Gravity is present.

$$0=\mu\frac{\partial^2 v_x}{\partial y^2}+\rho g_x$$ (10 points total)

(d) Our boundary condition at the free interface is: $\frac{dv_x}{dy}(y=y_o)=0$.

Derive the shear rate profile, ( $\frac{dv_x}{dy}$ as a function of $y$.)

$$\mu\frac{\partial^2 v_x}{\partial y^2}=-\rho g_x$$

$$\mu\frac{d}{dy}\left(\frac{dv_x}{dy}\right)=-\rho g_x$$

$$d\left(\frac{dv_x}{dy}\right)=-\frac{\rho g_x}{\mu}dy$$

$$\int_{\left.\frac{dv_x}{dy}\right|_y}^{\left.\frac{dv_x}{dy}\right|_{y_o}} d\left(\frac{dv_x}{dy}\right) = \int_y^{y_o} -\frac{\rho g_x}{\mu} dy$$

$$\left.\frac{dv_x}{dy}\right|_{y_o} - \left.\frac{dv_x}{dy}\right|_y = -\frac{\rho g_x}{\mu}(y_o - y)$$

$$0 - \left.\frac{dv_x}{dy}\right|_y = -\frac{\rho g_x}{\mu}(y_o - y)$$

$$\frac{dv_x}{dy} = \frac{\rho g_x}{\mu}(y_o - y)$$ (2 pts)

This is the shear rate as a function of y-position. It is a linear function of y. It is zero at the free interface, when $y = y_o$. It is maximized at the wall, where y = 0.

(e: extra-credit) Our boundary condition at the wall is: $v_x(y = 0) = 0$

Derive the velocity profile, ( $v_x$ as a function of $y$ .)

$$\frac{dv_x}{dy} = \frac{\rho g_x}{\mu}(y_o - y)$$

$$dv_x = \frac{\rho g_x}{\mu}(y_o - y)dy$$

$$\int_{\left.v_x\right|_{y=0}}^{\left.v_x\right|_y} dv_x = \int_{y=0}^{y} \frac{\rho g_x}{\mu}(y_o - y)dy$$

$$v_x - v_x(y = 0) = \frac{\rho g_x}{\mu}\left(y_o y - \frac{y^2}{2}\right)_0^y$$

$$v_x - 0 = \frac{\rho g_x}{\mu}\left(y_o y - \frac{y^2}{2}\right)$$

$$v_x = \frac{\rho g_x}{\mu}\left( y_o y - \frac{y^2}{2} \right) \qquad \text{(2 pts)}$$

The velocity profile is parabolic. The velocity is zero at the wall, where y = 0. The velocity has a maximum at the free interface, where $y = y_o$.